THE JOURNAL OF THE
ASTRONAUTICAL SCIENCES

VOLUME VIII, NO. 2 SUMMER 1961

CONTENTS

PUBLISHED BY THE AMERICAN ASTRONAUTICAL SOCIETY, INC.

THE AMERICAN ASTRONAUTICAL SOCIETY, INC.

516 Fifth Avenue, New York 36, New York, U.S.A.

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The
Journal of the Astronautical Sciences

Director of Publications, George R. Arthur
Editor, Robert M. L. Baker, Jr.
Managing Editor, George J. Clark

Published quarterly by the AMERICAN ASTRONAUTICAL SOCIETY, INC. at 428 E. Preston Street, Baltimore 2, Maryland

Address all Journal correspondence to Box 24721, Los Angeles 24, Calif.

Subscription Rates: One year $5.00; foreign $6.00; single copy $1.25. The Journal is published quarterly.

Second-class postage paid at Baltimore, Maryland.

An Integrated Approach to the Determination and Selection of Lunar Trajectories

Walter C. Nelson*

Abstract

An analytic approach to the determination and selection of ballistic lunar trajectories based on the restricted three-body problem is presented. A technique for relating a geocentric ellipse to a selenocentric hyperbola is shown. The technique leads to inter-relationships of major design areas including guidance and control, propulsion, tracking and communications, and logistics. Particular emphasis is placed on relating vehicle energy to selenographic coordinates for lunar landing and takeoff or for entering and leaving lunar orbits from which the "corridor-ring" concept is developed. In addition an analysis is conducted that defines the optimum point for entering and leaving lunar orbits or for lunar landing and takeoff.

Introduction

An approach to ballistic lunar trajectories using Keplerian orbits as a basic ingredient is presented. A geocentric ellipse with earth radius as perigee is matched geometrically and dynamically to the asymptote of a selenocentric hyperbola. For a lunar landing this hyperbola intersects the lunar surface at the landing site and for transferring to a lunar orbit, the point of transfer is at periselenum. A proper match of these Keplerian orbits is accomplished with the aid of the energy equation. The energy equation can be applied to translunar vehicles to determine the periselenum velocity decrement required for landing or orbiting. By a reverse process the velocity increment needed for transearth flight can be determined. The fact that the moon moves about 3500 ft/sec relative to earth leads to positive energy selenocentric vehicle orbits from which entry and exit corridors are derived.

Corridors are defined by the volume swept out when rotating a segment of a hyperbola about an axis parallel to the asymptote and passing through the moon's center. The locus of the periselenum point for a given positive energy lunar orbit determines rings for entry to or exit from a closed lunar orbit when periselenum is greater than the moon radius. Lunar orbits with the same positive energy are used to define rings for lunar landing and takeoff when periselenum is less than a moon radius. Rings are selenographically fixed if the small angular motion of the moon relative to the earth-moon line is neglected. The direction of the corridor-ring axis which becomes fixed when time of flight and trajectory inclination are specified defines the avenues for a vehicle approaching or departing the moon.

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In addition to the corridor-ring concept, an analysis is conducted showing that the optimum point along a hyperbolic orbit for entering or leaving a circular lunar orbit is at periselenum.

The ring-corridor concept is an aid to the lunar trajectory problem that provides some simple tools leading to an appropriate selection of ballistic trajectories for a given mission and in addition provides an integrated outlook to problems relating guidance and control, propulsion, communication and tracking, and logistics.

Analytical Model of the Earth Moon System

An approach to lunar trajectories using Keplerian orbits as a basic ingredient is presented. A geocentric ellipse with earth radius as perigee is matched geometrically and dynamically to the asymptote of a selenocentric hyperbola. The matching of the ellipse and hyperbola provides an analytical technique for obtaining initial conditions for ballistic translunar and transearth flight. With the initial geocentric or selenocentric velocity and the corresponding geographic or selenographic coordinates, a high speed computer can then be used to "zero in" on a precision trajectory. The precision trajectory is the result of integrating the vehicle motion in some appropriate mathematical model of the earth-moon system.

The analytical model of the earth-moon system used for the precision trajectory calculations consists of the earth and moon revolving in elliptic orbits about the barycenter with the barycenter in turn moving in an elliptic orbit about the sun. The moon is assumed to keep the same side facing the earth at all times.

The moon's orbit can be described relative to a directionally fixed coordinate system with origin at the center of the earth by two geometrical parameters

$$\text{eccentricity } e_m = 0.0549005$$

$$\text{semi latus rectum } p_m = 1.2585193 \times 10^9 \text{ ft.}$$

The mass characteristics of the earth and moon corresponding to the eccentricity and semi latus rectum can be represented by the respective products of their masses and Newton's gravitational constant

$$\mu_e = 1.40769 \times 10^{16} \text{ ft.}^3/\text{sec}^2$$

$$\mu_m = 1.7299 \times 10^{14} \text{ ft.}^3/\text{sec}^2.$$

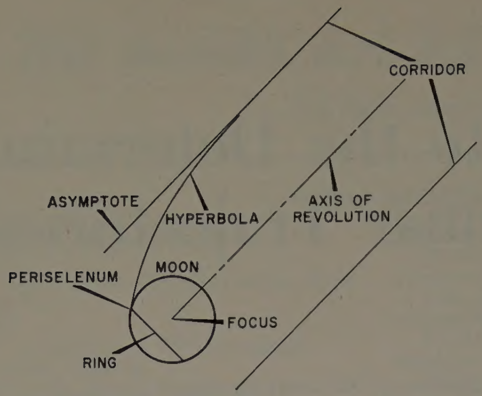

FIG. 1. Segment of hyperbola used to generate corridors and rings.

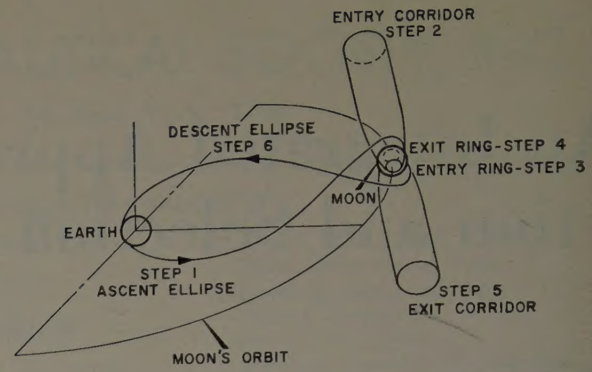

FIG. 2. Illustration of the six steps for orbiting or landing on the moon and returning to earth.

The radii of the earth and moon are assumed to be:

$$r_e = 20.92 \times 10^6 \text{ ft.}$$

$$r_m = 5.7024 \times 10^6 \text{ ft.},$$

respectively.

The constants used to represent the moon's orbit and the mass characteristics of the earth and moon are regarded as being typical but not necessarily consistent with the actual values.

The foregoing mathematical model is used to generate ballistic trajectories for translunar and transearth flight. The individual mass characteristics of the earth and moon and the geometrical arrangement of the earth and moon are in turn used in Keplerian orbit mechanics to estimate initial conditions for the trajectory integration.

Six Steps for Orbiting the Moon

A trip to the moon incorporating lunar orbiting can be visualized as taking place in six steps. Assume a trajectory coplanar with the lunar orbit plane.

1. Injection on to a geocentric ellipse
2. Transfer to a selenocentric hyperbola—"corridor"
3. Enter lunar orbit or land on the moon—"ring"
4. Takeoff from the moon or leave lunar orbit—"ring"
5. Injection on to a selenocentric hyperbola—"corridor"
6. Transfer to a geocentric ellipse

The primary idea contained in these steps is in the corridor-ring concept. The surface of the corridor is formed by rotating a segment of a hyperbola about an axis passing through the center of the moon, one asymptote being parallel to the axis of rotation. The center of the moon is assumed to be at the focus. The locus of the periselenum point on the hyperbola forms the ring on the surface of the moon or on the surface of a sphere concentric to the lunar sphere. The method for generating the corridor and ring is illustrated in Figure 1.

The geocentric ellipse of step 1 has a perigee near the surface of the earth with apogee extending to the near side of the moon or beyond the moon. A vehicle traveling to the moon moves away from the earth—see Figure 2—on this ellipse. When it has reached the vicinity of the moon's orbit the vehicle velocity will be low in relation to the moon's velocity. This in effect coupled with the gravitational attraction of the moon, transfers the vehicle on to a selenocentric hyperbola placing the vehicle on the surface of a corridor. As the moon moves toward a collision with the vehicle the vehicle is in effect moving along the surface of the corridor toward periselenum. At periselenum the vehicle is on the entry ring and application of an appropriate impulse puts it in a lunar orbit. Orbiting continues until return to earth is desired.

In general the point for entry of the lunar orbit will not be coincident with point for exit. However, once during each circumnavigation the vehicle passes over a point on an exit ring where it is suitable for initiating escape to return to earth. Then application of the appropriate impulse causes the vehicle to move away from the moon on a selenocentric hyperbola. Application of the escape impulse can be regarded as permitting the moon to move away from the vehicle which tends to maintain a fixed position relative to the earth. As the distance between the moon and vehicle increases, the vehicle velocity decreases and it transfers on to a geocentric ellipse and descends to earth. Perigee is assumed to be at the earth surface.

A quantitative description of the foregoing lunar mission can be formulated using the mechanics of Keplerian orbits. To do this a backward approach will be used and it will be assumed that the vehicle is in a lunar orbit and it is desired to bring it to earth ballistically. Two assumptions will be made concerning the lunar orbit for purposes of illustration, but these assumptions need not apply generally. First assume that the orbit is circular at zero altitude. This means the orbit velocity is given by $(\mu_m/r_m)^{1/2}$. Next the line of nodes formed by the intersection of orbit plane and the lunar orbit plane is perpendicular to the earth-moon line. This line is coincident with the velocity vector of the moon when the moon is at apogee and perigee.

Ballistic return to earth cannot be accomplished by providing the vehicle with a velocity only sufficient to

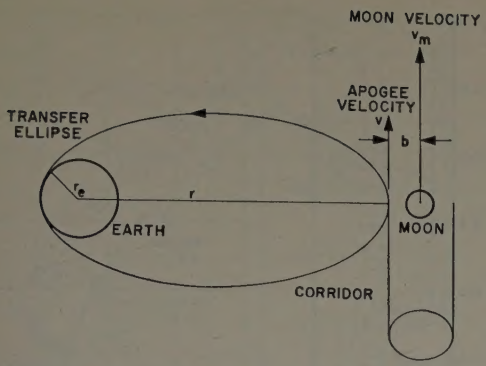

Fig. 3. Illustration of relation between geocentric ellipse and corridor for a minimum energy mission.

escape the moon $(2\mu_m/r_m)^{1/2}$. This can be seen examining the energy equation:

$$E = \frac{1}{2}v^2 - \frac{\mu}{r}. \tag{1}$$

Escape speed implies zero total energy. Therefore when the vehicle has escaped the moon and the distance from the moon is large, the relative velocity approaches zero. Since the velocity of the moon in a directionally fixed earth centered coordinate system is about 3500 ft/sec and essentially perpendicular to the earth-moon center line, it follows that the vehicle velocity relative to earth following lunar escape must be roughly the same as the moon's. That is, the velocity of the moon is inherent to the velocity of the vehicle. Consequently a geocentric vehicle orbit with dimensions similar to the dimensions of the moon's orbit will result. This problem can only be avoided by escaping the moon at hyper escape velocities.

The increment of velocity above escape needed to bring a vehicle to earth can be deduced from equation (1) by rewriting it in the form:

$$E = \frac{1}{2}(v_e + \Delta v)^2 - \frac{\mu}{r} = \frac{1}{2}v_\infty^2, \tag{2}$$

where Δv is the excess vehicle velocity above the escape velocity v_e and v_∞ is the velocity relative to the moon for large values of r. It follows that

$$\Delta v = (v_e^2 + v_\infty^2)^{1/2} - v_e\,. \tag{3}$$

This equation states that if the velocity at $r \rightarrow \infty$ is known, the amount Δv needed to be applied at the lunar surface to produce v_∞ can be determined.

A low energy transfer from the moon to earth can be used to illustrate the application of equation (3). Suppose it is desired to transfer a vehicle from a selenocentric escape orbit on to the apogee of a geocentric ellipse. The apogee velocity relative to earth is given by:

$$v_a = \left[\mu_e\left(\frac{2}{r} - \frac{1}{a}\right)\right]^{1/2}, \tag{4}$$

where r is the distance from the center of the earth to the surface of the corridor and

$$a = \tfrac{1}{2}(r + r_e), \tag{5}$$

where r_e is the earth radius—see Figure 3. It is desired that v_∞ be equal to the difference between the apogee velocity of the geocentric ellipse and the velocity of the moon.

$$v_\infty = v_m - v_a\,, \tag{6}$$

where v_m is the magnitude of the moon's velocity. The relation of v_∞ to the geometrical and dynamical properties of a selenocentric hyperbola with periselenum at the lunar surface can now be derived. From the Keplerian orbit equation

$$p = r_m(1 + e), \tag{7}$$

where p is the semi latus rectum of the hyperbola, e is the eccentricity, and r_m is the moon radius. Equations (3) through (7) can be used to derive expressions for the geometrical description of the escape hyperbola in terms of the required escape speed.

$$e = 2\left(\frac{v_e + \Delta v}{v_e}\right)^2 - 1 \tag{8}$$

$$p = 2\left(\frac{v_e + \Delta v}{v_e}\right)^2 r_m\,. \tag{9}$$

Having obtained the geometrical properties of the escape orbit the dimensions of the corridor and ring follow and these in turn are related to the velocity applied for escape. The corridor radius is as follows:

$$b = \frac{r_m}{\left[1 + \left(\frac{v_e}{v_e + \Delta v}\right)^2\right]^{1/2}}. \tag{10}$$

The quantity b is called the guidance parameter according to Mickelwait in reference 3. Relations between Δv, e, and p are given in Figure 4. The ring size is specified by the cone angle subtended at the center of the moon:

$$\rho = 180° - \cos^{-1}\left(-\frac{1}{e}\right). \tag{11}$$

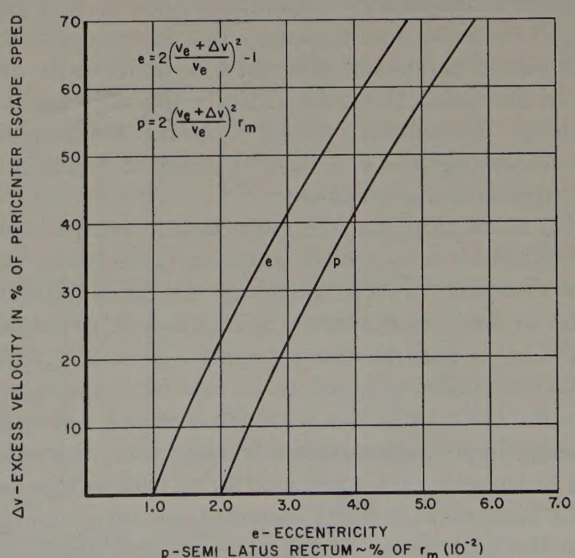

Fig. 4. Semi latus rectum and eccentricity of a hyperbola vs. excess velocity applied at pericenter.

TABLE 1

Comparison of the low energy mission obtained from Keplerian orbit calculations to integrated trajectory calculations

Parameter	Keplerian	Integrated Trajectory
Corridor radius (moon radii)	2.85	2.93
Ring size—ρ	39.5°	37.7°
Δv ft/sec	526	499
Geocentric ellipse eccentricity	.966	.967
Time of flight (days)	3.098 Apogee to perigee	4.531 Periselenum to perigee

Equation 10 shows that as the injection speed $v_e + \Delta v$ increases the corridor radius gets smaller and equation 11 indicates that the ring size increases because e increases with escape speed. Some limitations on Δv will be discussed in subsequent paragraphs.

Some specific results for the low energy mission have been tabulated in Table 1. The values tabulated under the integrated trajectory were obtained using the initial conditions from the trajectory computation in conjunction with equations (10) and (11), except for the geocentric ellipse eccentricity and time of flight. These were extracted directly from the computed trajectory. It is seen that the corridor radius is small in comparison to the distance to the moon. Because of this it can be shown that the Δv needed to orbit the moon or escape a lunar orbit on the corridor surface farthest from the earth will differ a negligible amount from the given value. This is because the apogee velocity of a geocentric elliptic orbit changes only a slight amount for changes of apogee over the dimensions of the corridor diameter, which in effect maintains v_∞ constant. On the basis of this result the rings and corridors are defined to be circular. It is of interest that subsequent trajectory computations showed that getting off the moon on a trajectory that moves across the back side of the moon, over the top, underneath, and across the visible face took virtually the same amount of energy. These results can be of interest to communications and logistics of a mission design in addition to the propulsion and guidance.

If it is assumed that the moon is in a circular orbit, the trajectory of a vehicle traveling to the moon can have symmetry with a trajectory that goes from the moon to earth, as illustrated in Figure 5. A round trip trajectory is plotted showing relative positions of the moon and vehicle as functions of time for a low energy mission. The coordinates are earth centered and directionally fixed. Instantaneous transfer from the entry ring to the exit ring is assumed along with application of the appropriate velocity increment for entry and exit from the lunar orbit. The inset shows the vehicle trajectory in the vicinity of the moon in a directionally fixed earth centered coordinate system.

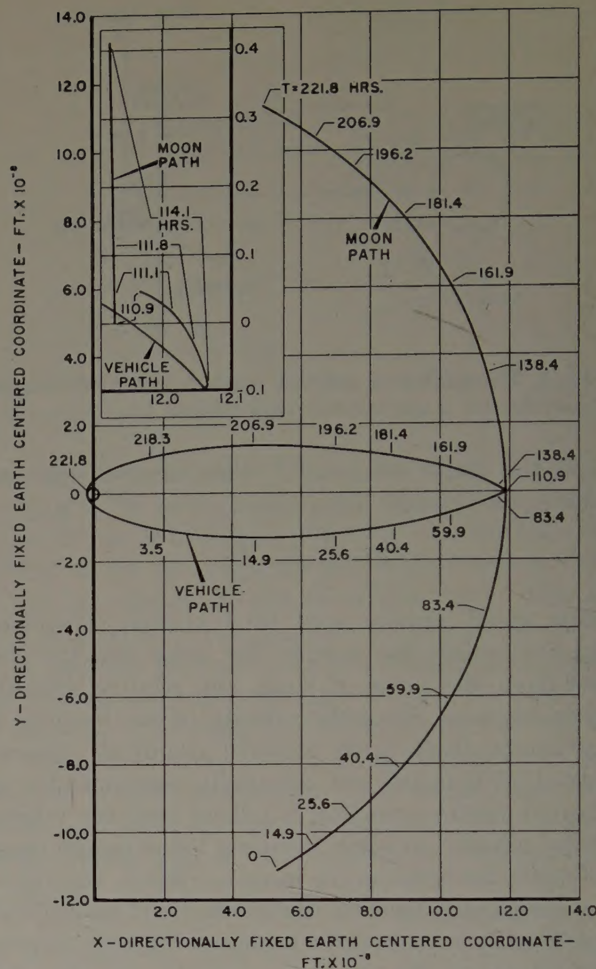

FIG. 5. Low energy round trip to the moon. Trajectory coplanar to moon orbit plane.

Lunar Landing and Takeoff

Lunar trajectories applicable to missions incorporating landing and takeoff can be developed with the aid of the corridor-ring concept. It was indicated that the energy relative to the moon required for a moon to earth transfer on the surface of a given corridor is essentially constant. This was because the surface of the corridor represented the locus of asymptotes formed by rotating a given hyperbola about an axis of revolution through the center of the moon. Consider any selenocentric hyperbola with this same total energy relative to the moon but suppose periselenum is smaller, that is under the surface of the moon. This means a smaller diameter "corridor" can be generated with a hyperbola intersecting the lunar surface as shown in Figure 6. It can be deduced from Equation (3) that constant Δv is associated with any of these hyperbolas at the point of intersection with the lunar surface. This is because v_∞ was assumed to be constant and v_e is necessarily constant for all points on the lunar sphere. The distinction between any two given landing or takeoff points on the lunar sphere in relation to a given Δv is the angle at which the hyperbola intersects the lunar sphere. This angle can be called the launch or landing angle. Figure 7 shows how this angle is related to points on the lunar

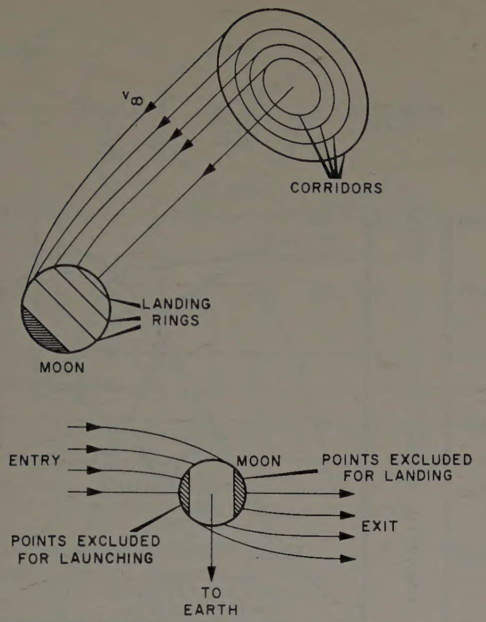

Fig. 6. Illustration of corridors and rings for lunar landing and takeoff.

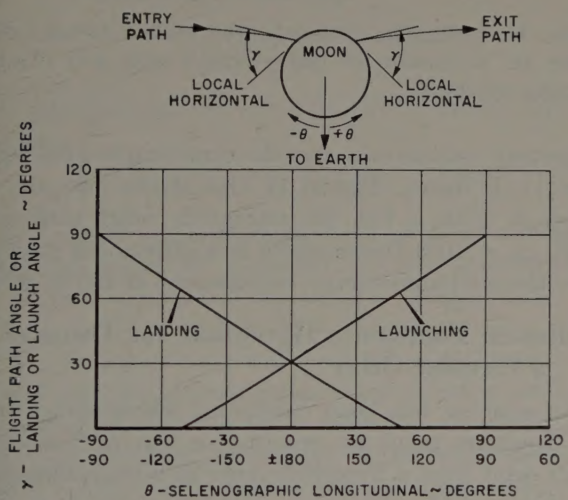

Fig. 7. Lunar landing and takeoff angle vs. selenographic longitude in the selenocentric vehicle orbit plane for the low energy mission.

equator for the low energy mission. Points on the lunar sphere that have the same landing and launch angle form circles concentric to the axis of the approach or exit corridor.

In general an approach or exit corridor is associated with a ring whose cone angle is less than 180°. Therefore, for a given corridor a majority of the lunar surface can be regarded as exterior to a given ring. And because constant values of v_∞ and Δv are assumed to be associated with any approach or exit hyperbola interior to the corridor with asymptote parallel to the corridor axis, it follows that most of the lunar surface is available for either landing or takeoff using a single impulse.

Time of Flight

To this point discussion has centered on escaping a lunar orbit and transferring to a geocentric ellipse at apogee. This is regarded as a minimum energy transfer and since essentially the complete half of a geocentric ellipse is traversed the flight is regarded as taking maximum time. Because shorter flight times are of interest, particularly for manned missions involving solar flare threats, it will be shown how the corridor-ring concept can be broadened for reducing flight time. It happens that as time of flight decreases, the corridor axis tilts toward the earth moving the ring toward the invisible side of the moon. This can be seen quantitatively by noting that when leaving the moon the vector $\bar{v}_\infty$ should have a component directed toward the earth if flight time is to be shortened. Therefore equation (6) takes a vectorial representation:

$$\bar{v}_\infty = \bar{v}_m - \bar{v}, \tag{12}$$

where $\bar{v}$ is a velocity associated with some point below apogee on a geocentric ellipse as indicated in Figure 8

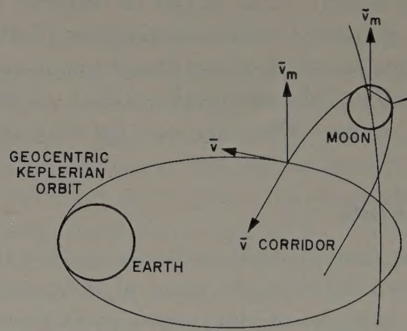

Fig. 8. Illustration of technique for estimating reduced flight times.

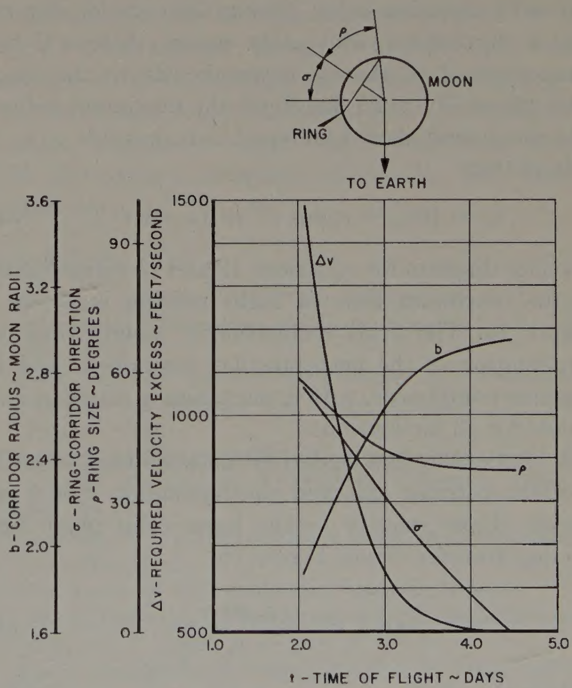

Fig. 9. Corridor radius
ring—corridor axis direction
ring size and required velocity increment vs. time of flight

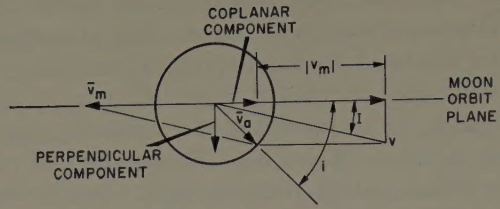

FIG. 10. Vector diagram illustrating vector relationships for entering inclined geocentric orbits on the low energy mission.

Using this approach, estimates for the changes in corridor and ring sizes were made along with changes in direction of the corridor-ring axis for missions with flight times less than 4.46 days. A high speed digital computer was then used iteratively to obtain curves of ring size, corridor size, direction of ring-corridor axis, and Δv for various flight times, Figure 9. The results show that fuel penalty for missions under 3.5 days increases very rapidly. Also, it can be deduced from the curve that a ballistic circumnavigation of the moon that intersects the earth-moon line at one moon radius, from the center of the moon takes about 5.2 days from perigee to perigee. The ring size for this mission is $\rho = 48°$.

Inclined Orbits

To reduce time of flight, it was seen that the corridor-ring axis rotated in the plane of the moon's orbit. Tilting the axis out of the moon's orbit plane causes the plane of the vehicle orbit relative to the earth to be inclined to the moon's orbit plane. This is important because it permits selection of trajectories that avoid the earth radiation belts. Tilting the corridor-ring axis out of the moon's orbit plane means there will be a component of $\bar{v}_\infty$ that is perpendicular to the moon-orbit plane. For a 4.5 day flight the maximum value of this component should be equal in magnitude to v_a. It follows that

$$v_\infty = [(v_m - v_a \cos i)^2 + (v_a \sin i)^2]^{1/2}. \quad (13)$$

A vector diagram for equations 12 and 13 corresponding to the maximum time of flight mission is shown in Figure 10. The orbit inclination i, results from the combination of the perpendicular component and the coplanar component, with v_a maintaining constant magnitude for all inclinations.

A relationship can be derived between the inclination I of the corridor axis and the inclination i of a geocentric ellipse relative to the lunar orbit plane for a 4.5 day transfer. From Figure 10

$$\tan I = \frac{v_a \sin i}{v_m - v_a \cos i}. \quad (14)$$

Tilting the corridor axis about 10° produces an inclination of 90°. Inclinations greater than 90° are associated with retrograde geocentric orbits. Figure 11 gives results for the relation between I and i as obtained from trajectory calculations and the comparison with equation 14 is shown. Figure 11 also shows how the Δv increases with i. For the retrograde orbit with $i = 180°$, $\Delta v = 1031$ ft/sec which is a substantial increase over the minimum energy requirement of 499 ft/sec.

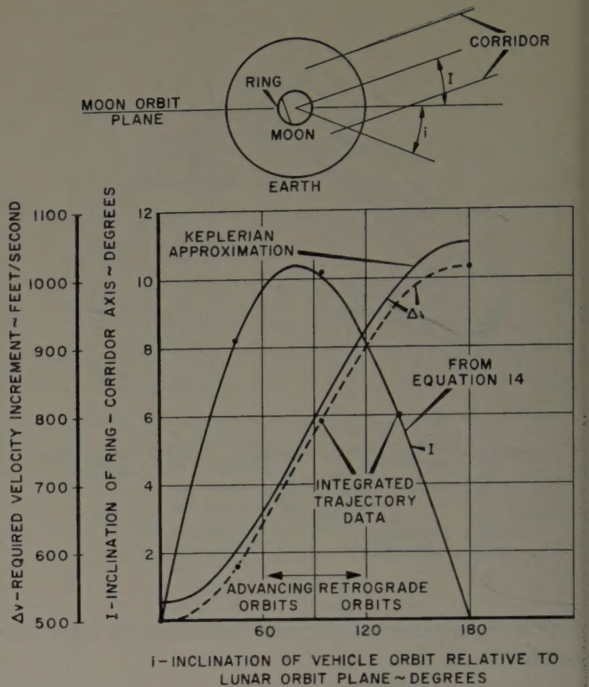

FIG. 11. Inclination of vehicle orbit relative to lunar orbit plane vs. inclination of ring—corridor axis and required velocity increment.

Optimum Point on a Hyperbola for Transfer to a Circular Orbit

An analysis has been conducted which shows that the optimum point for entering or leaving a circular lunar orbit from a hyperbolic approach trajectory is at periselenum. On the basis of the analysis it is concluded that lunar missions incorporating circular lunar orbits should be designed around a trajectory that has periselenum at the altitude of the desired orbit.

Suppose a vehicle is approaching the moon on a hyperbolic path as indicated in Figure 12 and transfer to a circular orbit is desired. This means at some point along the hyperbolic path the total energy must be decreased to a negative value. The total energy per unit mass is

$$E = \frac{1}{2}v^2 - \frac{\mu}{r}. \quad (15)$$

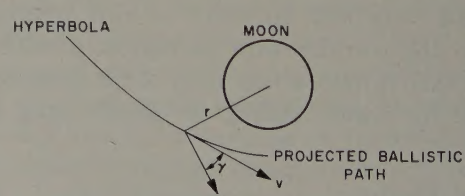

FIG. 12. Illustration of technique for entering a circular lunar orbit from a hyperbolic orbit.

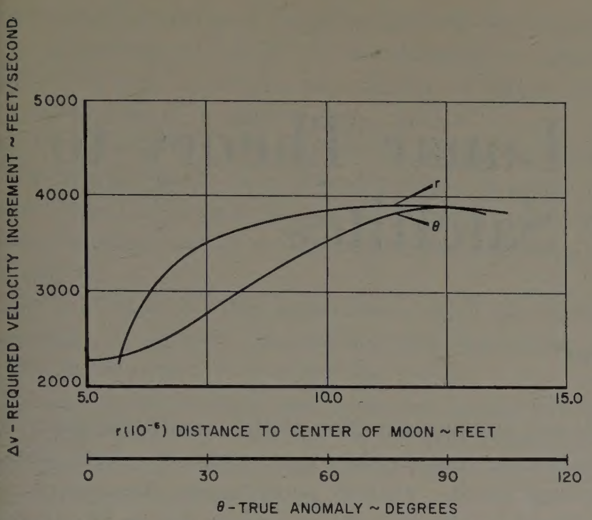

FIG. 13. Velocity increment required to enter circular orbit from a selenocentric parabola vs. distance to moon center and true anomaly.

The circular orbit velocity is

$$v_c = \sqrt{\frac{\mu}{r}}. \tag{16}$$

The increment of velocity required to enter a circular orbit is

$$\delta v = [(v \sin \gamma)^2 + (v \cos \gamma - v_c)^2]^{1/2}, \tag{17}$$

where γ is the vehicle flight path angle at any point along the hyperbola. Equation (17) can be rewritten:

$$\delta v = \left[2E + \frac{3\mu}{r} - \frac{2h}{r}\sqrt{\frac{\mu}{r}}\right]^{1/2}. \tag{18}$$

Where h is the angular momentum per unit mass. To determine if there is some point on the hyperbolic path that requires a minimum or maximum δv to get into a circular orbit, differentiate δv with respect to r and set it to zero. It follows that

$$r = \frac{h^2}{\mu} \quad \text{for} \quad \frac{d}{dr}(\delta v) = 0. \tag{19}$$

Since h^2/μ is the semi latus rectum of an orbit, equation (19) states that for a given hyperbolic orbit the energy needed to go into circular orbit is a maximum or minimum when the vehicle is at the semi latus rectum point or when it is at infinity. A curve for the required velocity increment versus distance from the center of the moon is given in Figure 13 assuming parabolic motion with periselenum at the moon surface. It is illustrated that the velocity increment at the semi latus rectum point is a maximum. The required velocity increment at periselenum is a minimum although it is not a mathematical minimum. It is concluded that the best point to enter circular orbit from an excess energy orbit is at pericenter.

If the class of all excess energy orbits having the same pericenter is considered, a curve of the velocity increment at pericenter in excess of circular orbit speed can be prepared and this is presented in Figure 14. Since all the excess energy orbits have the same pericenter distance it is convenient to use the eccentricity or semi latus rectum as the abcissa. δv is given in % of the circular orbit velocity. For purposes of comparison the velocity increment required for entering circular orbit at the semi latus rectum point is also presented. It is apparent that a substantial fuel saving is to be realized by selecting the pericenter point along a hyperbolic orbit for entry departure from a circular orbit.

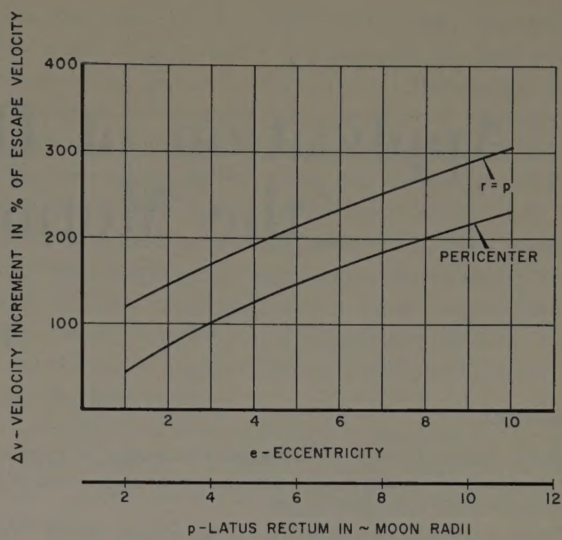

FIG. 14. Velocity increment required to inject into circular lunar orbit at pericenter and at $r = p$.

Acknowledgements

The author wishes to express his gratitude to Dr. Wilhelm A. Elfers and Mr. Ernest E. Loft for their helpful criticisms of the manuscript and to Mr. Walter Lee and Miss Joy Floyd for their assistance in preparation of the data. Trajectory computations were made with an IBM 709 using a program written by Mr. Thomas Englar.

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Application of Hill's Lunar Theory to the Motion of Satellites

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Abstract

Hill's Lunar Theory of Celestial Mechanics is applied to the problem of establishing a permanent artificial satellite on a periodic orbit around a planet.

Using a method developed by C. L. Siegel, in his "Vorlesungen über Himmelsmechanik," the Hill's equations of the Lunar Theory are solved to obtain the coordinates of the periodic trajectory as Fourier series of the time with respect to a rotating system of reference. A recurrent procedure is obtained for evaluating the coefficients of the series in terms of the period of revolution. The Jacobi constant of the motion is also expressed as an infinite power series of the period. The convergence of such expansions can be ascertained for small values of the period. A numerical example for a satellite of Venus is furnished.

An error analysis is undertaken by studying solutions of Hill's equations lying in a neighborhood of a periodic orbit and corresponding to the same value of the total energy. The coordinates of such neighboring trajectories are determined as isoenergetic displacements referred to the intrinsic reference formed by the tangent and normal lines at the various points of a periodic orbit. This procedure leads to a differential equation of the Mathieu type whose solution is obtained as a series expansion valid for small values of a parameter.

Introduction

The problem of establishing a permanent artificial satellite around a planet of our solar system (say Venus or Mars) is becoming of actual concern. For many reasons, among which are the stability and tracking questions, a periodic trajectory is the ideal solution.

The trajectory that takes the ballistic missile from the neighborhood of the Earth (of the order of a couple hundred miles above the Earth's surface) to the vicinity of the planet belongs decidedly to the restricted three-body problem.

Extensive numerical studies of such trajectories for lunar purposes, whose results are however applicable to interplanetary flight, have been undertaken by Russian scientists see Ref. [9], but none of these trajectories achieves a permanent periodic orbit. It appears, therefore, due to the present limited knowledge, that the missile should have a retrorocket which, by slowing it down, would ensure its permanent capture by the planet.

However, even within the sphere of action† of the planet, an approximation of the missile trajectory by Keplerian motion does not seem to yield accura enough results: various perturbation procedures mu be used to obtain, with a greater degree of certaint the position and velocity of the satellite.

The Lunar Theory of Celestial Mechanics develope by G. W. Hill, at the end of the last century, as a pa ticular case of the restricted three-body problem ar later on elaborated upon by E. W. Brown, appea today as the most precise treatment of any perturbatic procedure. The latest developments, due to C. Siegel, even allow the set up of recurrent procedur which are highly desirable for numerical calculations b means of electronic computers in order to evaluate tl coefficients appearing in the series representation of tl trajectories. The present paper elaborates such pr cedures and applies them to the case of periodic orbit

In the restricted three-body problem, we consid two bodies of finite masses moving according to circul Keplerian motions around their centroid and study tl motion of a third body of infinitesimal mass in tl plane of the circular motions. The third body is su posed to be attracted by the two finite bodies accordir to Newton's law of gravitation, without affecting the relative motion because of its infinitesimal mass.

In the Lunar Theory, we suppose that one of the tw finite bodies has a mass considerably smaller than tl other, as might be the case of Venus or Mars wit respect to the Sun, and study the planar trajectories a third body of infinitesimal mass which encircle onl the body of small mass.

The periodic orbits of the Lunar Theory can b shown to take into consideration the main solar pe turbation on the circular orbits of a two-body proble and play the role of the so-called "intermediat orbits in perturbation theory.

In the technical problem of establishing a period orbit, three questions belong to the domain of celesti mechanics:

1. The value of the Jacobi constant, in terms of tl period of the orbit to be achieved, will allow us ascertain the desired velocity at the time of retrorock burnout.

2. Series expansions for determining the position ar velocity of the missile along a periodic trajectory; tl

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† For the definition of sphere of action of a planet see [9].

is necessary for tracking purposes and will also enable us to determine values of the acceleration to which the satellite will be subject, information of actual concern in the case of a manned space station.

3. Procedures to study trajectories, which can occur in the neighborhood of a periodic orbit, in order to realize what will happen in case the periodic trajectory cannot be achieved due to uncertainties in some engine parameters.

The first part of this paper deals with the series expansions of periodic trajectories and expression of the Jacobi constant in terms of the period.

The assumptions which constitute the Lunar Theory reduce the equations of motion for the restricted three-body problem to a particular set of differential equations known as Hill's equations.

General existence theorems for solutions of differential equations assure that under certain conditions satisfied by Hill's equations, the solutions can be expressed in terms of infinite power series in the variables, within a certain interval of convergence. The procedure consists in assuming, as solutions of Hill's equations, appropriately chosen infinite power series in two variables with undetermined coefficients; the two variables are moreover considered solutions of an auxiliary differential system which is known to possess periodic solutions. In trying to satisfy Hill's equations, a recurrent procedure arises whereby the coefficients of each power can be uniquely determined, once the coefficients of lower powers are known.

The absolute convergence of such expansions can be established by a "majorant" method which consists in comparing the trial series with series having larger coefficients and whose convergence can be easily ascertained.

In replacing the two variables by particular periodic solutions of the auxiliary system, we obtain periodic solutions for Hill's system with the same value of the period.

By rearranging the various terms of the series, these periodic solutions ultimately appear as Fourier series of the time; the Fourier coefficients being infinite power series of the period, convergent for small values of the period.

The recurrent procedure mentioned before enables us to determine as many terms as might be required in the expansions of the coefficients of the Fourier series. Some of these terms have been obtained by hand computation.

A similar procedure is used to express the Jacobi constant of the motion as a convergent infinite series of the period.

A numerical example for a satellite of Venus has been worked out; for this purpose, a program on the IBM 704 has been used to determine the coefficients of the Fourier series up to the twelfth power of the period. An idea of such numerical results is furnished by Table 1.

The last part of the paper deals with trajectories lying in the neighborhood of a periodic one and which correspond to the same value of the total energy. This study can be interpreted either as an error analysis for periodic trajectories or might be thought of as furnishing the secondary perturbation effects due to the Sun.

Referred to the intrinsic reference constituted by the tangent and normal lines at points of a fixed periodic orbit, these neighboring trajectories can be specified by means of their tangential and normal displacements.

The normal isoenergetic displacement satisfies, for a certain degree of approximation, a differential equation of the Mathieu type.

A modification of a well-known method for solving such equations by an infinite cosine series is outlined which is very convenient for rapid calculation of the coefficients.

The parameter, which appears in the argument of the cosine functions, is also determined by an elementary method which, although mathematically rigorous, circumvents the question of infinite determinants. The paper is meant to be analytical in character and while a brief survey of the Lunar Theory must necessarily be exposed for reasons of clarity, it emphasizes primarily, within the framework of mathematical rigour, new numerical methods for determining the essential parameters of motion.

Periodic Orbits

We shall denote the planet by P_1 and the Sun by P_2, with masses m_1 and m_2 respectively. These two celestial bodies will be supposed spheres, homogeneous in concentric layers. Neglecting the eccentricity of the planet's orbit around the Sun (in the case of Venus this eccentricity is approximately 0.00681, the smallest in the solar system), we shall assume P_1 as describing a circular, uniform path around P_2 with sidereal period τ^*. The missile will be denoted by P_3, with zero mass ($m_3 = 0$) and will be supposed to move in the plane of the circular motion, subject to the Newtonian attractions of the planet and Sun without, however, disturbing the Keplerian motion of these two bodies. The study of the motion of P_3 (infinitesimal body) constitutes the so-called restricted three-body problem.

The following fundamental units are chosen:

R_{12} = distance (P_1P_2) as unit of length,

$m_1 + m_2$ as unit of mass,

$\frac{\tau^*}{2\pi}$ as unit of time.

It follows then, that

$\mu = \frac{m_1}{m_1 + m_2}$, with $0 < \mu < 1$ will represent the mass of P_1,

$1 - \mu = \frac{m_2}{m_1 + m_2}$ will represent the mass of P_2,

$\omega^* = \frac{2\pi}{\tau^*}$ will be the unit of angular velocity and the

universal constant of gravitation

$G = \dfrac{(\omega^*)^2 R_{12}^3}{m_1 + m_2}$ will have the value one.

We consider, next, a Cartesian orthogonal frame in the plane of the circular motion with origin at the planet $P_1(0; 0)$ and rotating with constant angular velocity equal to one.

With respect to such reference, the Sun will occupy the fixed position $P_2(-1; 0)$ and the differential equations for the motion of $P_3(x; y)$ are given by (see Ref. [8] page 347, Ref. [6] page 291–95):

$$\begin{aligned} \ddot{x} &= 2\dot{y} + x + (1 - \mu) + F_x \\ \ddot{y} &= -2\dot{x} + y + F_y \\ F(x; y) &\equiv \frac{1 - \mu}{[(x + 1)^2 + y^2]^{1/2}} + \frac{\mu}{(x^2 + y^2)^{1/2}}, \end{aligned} \tag{1}$$

where dots denote derivatives with respect to the time, and letters used as subscripts, partial derivatives with respect to the same variables.

Introducing the complex, conjugate variables $(u; v)$ by means of the regular transformation of coordinates

$$\begin{aligned} u &= \frac{1}{\mu^{1/3}}(x + iy) \\ v &= \frac{1}{\mu^{1/3}}(x - iy); \quad (i^2 = -1), \end{aligned} \tag{2}$$

the system (1) will be taken into

$$\begin{aligned} \ddot{u} &= -2i\dot{u} + H_v \\ \ddot{v} &= 2i\dot{v} + H_u \\ H(u; v) &\equiv uv + \frac{2}{(uv)^{1/2}} + \frac{1 - \mu}{\mu^{1/3}}(u + v) \\ &\quad + \frac{2(1 - \mu)}{\mu^{2/3}(1 + u\mu^{1/3})^{1/2}(1 + v\mu^{1/3})^{1/2}}. \end{aligned} \tag{3}$$

If the mass μ of P_1 is so small compared to the mass $1 - \mu$ of P_2 that we can neglect the positive powers of $\mu^{1/3}$ in the expansion of $H(u; v)$, then we get the Lunar Theory provided we limit ourselves to the study of trajectories surrounding only the planet P_1. This assumption is certainly plausible in the case of Mars and Venus. For Venus $\mu = 1/408{,}001$ approximately. We have then the following:

$$H(u; v) \equiv \frac{2}{\mu^{2/3}} + \frac{3}{4}(u + v)^2 + \frac{2}{(uv)^{1/2}}$$

plus positive powers of $\mu^{1/3}$, hence we get Hill's differential equations:

$$\begin{aligned} \ddot{u} &= -2i\dot{u} + \frac{3}{2}(u + v) - \frac{u}{(uv)^{3/2}} \\ \ddot{v} &= 2i\dot{v} + \frac{3}{2}(u + v) - \frac{v}{(uv)^{3/2}}. \end{aligned} \tag{4}$$

We shall seek solutions of Hill's equations in the form of infinite power series in two variables with undetermined coefficients.

For this purpose let $U(t)$ and $V(t)$ be periodic solutions of the auxiliary differential system

$$\begin{aligned} \dot{U} &= \alpha U; \qquad \dot{V} = -\alpha V \\ \alpha &= \pm \frac{i}{(UV)^3}; \qquad (i^2 = -1), \end{aligned} \tag{5}$$

and consider the expressions:

$$\begin{aligned} W_{j;k} &\equiv U^{3j+4k} V^{3j-4k} \\ W_{j;-k} &= U^{3j-4k} V^{3j+4k}. \end{aligned} \tag{6}$$

Here j is any positive integer and k any integer (positive, zero or negative) satisfying the inequality

$$3j \geqq 4\,|k|,$$

thus, every exponent in (6) shall be positive and moreover

$$W_{j;k} W_{g;h} = W_{j+g;k+h}. \tag{6'}$$

The solutions we propose to consider are:

$$\begin{aligned} u &= U^4(1 + \textstyle\sum_{jk} A_{j;k} W_{j;k}) \\ v &= V^4(1 + \textstyle\sum_{jk} A_{j;k} W_{j;-k}), \end{aligned} \tag{7}$$

the $A_{j;k}$'s being the coefficients to be determined and where the summations shall be considered as double summations, the index j running from one to $+\infty$ and for each j we must consider the sum of the finite number of terms corresponding to k such that $4\,|\,k\,| \leqq 3j$; e.g.

$$\textstyle\sum_{jk} A_{j;k} W_{j;k} = A_{1;0} W_{1;0} + (A_{2;-1} W_{2;-1} + A_{2;0} W_{2;0} + A_{2;1} W_{2;1}) + \cdots$$

By formal manipulation of the series (7), we shall find the conditions to be satisfied by their coefficients so that the (7) be solutions of the equations (4). At a later stage, we shall discuss the absolute convergence of such expressions thus justifying the formal operations.

For sake of conciseness, we introduce the notation:

$$\begin{aligned} A &\equiv -\textstyle\sum_{jk} A_{j;k} W_{j;k} \\ B &\equiv -\textstyle\sum_{jk} A_{j;k} W_{j;-k} \\ C &\equiv (1 - A)^{-1/2}(1 - B)^{-3/2} - 1 - \tfrac{1}{2}A - \tfrac{3}{2}B. \end{aligned} \tag{8}$$

By formal expansions of $(1 - A)^{-1/2}$ and $(1 - B)^{-3/2}$ into power series of A and B, using the first two expressions (8) and (6'), C can ultimately be expressed as a double summation in the $W_{j;k}$'s. Differentiating (7) termwise with respect to the time and eliminating $\dot{U}$, $\dot{V}$ through (5), we get:

$$\begin{aligned} \dot{u} &= 4\alpha U^4[1 + \textstyle\sum_{jk}(2k + 1) A_{j;k} W_{j;k}] \\ \dot{v} &= -4\alpha V^4[1 + \textstyle\sum_{jk}(2k + 1) A_{j;k} W_{j;-k}] \\ \ddot{u} &= (4\alpha)^2 U^4[1 + \textstyle\sum_{jk}(2k + 1)^2 A_{j;k} W_{j;k}] \\ \ddot{v} &= (4\alpha)^2 V^4[1 + \textstyle\sum_{jk}(2k + 1)^2 A_{j;k} W_{j;-k}]. \end{aligned} \tag{7'}$$

Multiplying the two Hill's equations by U^2V^6 and U^6V^2 respectively, replacing in them the (7) and (7')

for the unknown functions and their derivatives, and making use of the notations (8), we find that the following expression:

$$\begin{aligned}[\textstyle\sum_{jk} (2k+1)^2 A_{j;k} W_{j;k}] - \tfrac{1}{2}A - \tfrac{3}{2}B \\ = C \mp 2[W_{1;0} + \textstyle\sum_{jk} (2k+1) A_{j;k} W_{j+1;k}] \quad (9) \\ - \tfrac{3}{2}W_{2;0}(1-A) - \tfrac{3}{2}W_{2;-1}(1-B),\end{aligned}$$

must be an identity in the $W_{j;k}$'s.

If we denote by $C_{j;k}$ the coefficient of $W_{j;k}$ in the expression appearing in the right-hand side of (9), it is possible to realize that $C_{j;k}$ turns out to be a polynomial (i.e., a finite expression) in the $A_{r;s}$'s with $r < j$.

This property allows us to write from (9) the recurrent linear system:

$$(k \neq 0) \begin{Bmatrix} [\tfrac{1}{2} + (1+2k)^2] A_{j;k} + \tfrac{3}{2} A_{j;-k} = C_{j;k} \\ \tfrac{3}{2} A_{j;k} + [\tfrac{1}{2} + (1-2k)^2] A_{j;-k} = C_{j;-k} \\ A_{j;0} = \tfrac{1}{3} C_{j;0} \end{Bmatrix}, \tag{10}$$

whose solution is easily found to be:

$$\begin{aligned} A_{j;k} &= \frac{\frac{1}{2} + (1-2k)^2}{4k^2(4k^2-1)} C_{j;k} \\ &\quad - \frac{\frac{3}{2}}{4k^2(4k^2-1)} C_{j;-k}; \quad (k \neq 0) \qquad (11) \\ A_{j;0} &= \tfrac{1}{3} C_{j;0}. \end{aligned}$$

It is possible to ascertain that the series in question, whose coefficients satisfy the linear system (10) (being thus formal solutions of Hill's equations), are absolutely convergent for small values of $|U|$ and $|V|$. The proof is based on a "majorant" procedure and is omitted from this paper; an idea of such proof can be obtained from [5] pages 89–92.

Among the solutions of the auxiliary system (5), we shall choose the periodic one

$$\begin{aligned} U &= \sigma^{1/2} \exp(\alpha t); \qquad V = \sigma^{1/2} \exp(-\alpha t) \\ 4\alpha &= \pm \frac{i}{\sigma^3}, \qquad (i^2 = -1), \end{aligned} \tag{5'}$$

where $\sigma^{1/2}$ is a real quantity.

Thus, if we introduce the notation:

$$\begin{aligned} A_0 &\equiv 1 + \sum_{1}^{\infty}{}_j A_{j;0} \sigma^{3j} \\ A_k &\equiv \textstyle\sum_j A_{j;k} \sigma^{3j} \\ &(k \neq 0,\ 3j \geqq 4|k|), \end{aligned} \tag{12}$$

Hill's equations will have the following exponential solutions:

$$\begin{aligned} u &= \sigma^2 \sum_{-\infty}^{\infty} A_k \exp\left(\pm \frac{2k+1}{\sigma^3} it\right) \\ v &= \sigma^2 \sum_{-\infty}^{\infty} A_k \exp\left(\mp \frac{2k+1}{\sigma^3} it\right). \end{aligned} \tag{13}$$

In terms of the Cartesian coordinates in the rotating system

$$x = \frac{1}{2}\mu^{1/3}(v+u); \qquad y = \frac{i}{2}\mu^{1/3}(v-u), \tag{14}$$

the solutions for the motion of the third body appear as infinite series of the Fourier type:

$$\begin{aligned} x &= \sigma^2 \mu^{1/3} \sum_{-\infty}^{\infty}{}_k A_k \cos\left(\frac{2k+1}{\sigma^3} t\right) \\ &\text{and} \qquad (15) \\ y &= \pm \sigma^2 \mu^{1/3} \sum_{-\infty}^{\infty}{}_k A_k \sin\left(\frac{2k+1}{\sigma^3} t\right). \end{aligned}$$

These formulae represent two families of periodic solutions for the motion of P_3 around P_1 because the period $\tau = 2\pi\sigma^3$ is an arbitrary parameter, within the region of convergence of the coefficients, and the double sign appearing in the second equation corresponds to the possibility of a direct or retrograde motion of P_3 around P_1 as compared to the circular motion of P_1 about P_2.

It is easy to realize that these periodic solutions are also symmetric with respect to both axes of the rotating frame so that only one-fourth of the trajectory need be studied.

The coefficient A_k of these series solutions, as defined by (12), are power series in the parameter σ^3, convergent for small values of the parameter and starting with terms in $\sigma^{4|k|}$ at least. By means of the recurrent procedure, illustrated above, it is possible to find as many terms as might be necessary in the expression for the A_k's even by hand computation.

The following approximations have been found by hand computation up to the fourth power of σ^3:

$$\begin{aligned} A_0 &= 1 - \tfrac{2}{3}\sigma^3 + \tfrac{7}{18}\sigma^6 - \tfrac{4}{81}\sigma^9 + \tfrac{19.565}{62.208}\sigma^{12} \\ A_1 &= \tfrac{3}{16}\sigma^6 + \tfrac{3}{8}\sigma^9 + \tfrac{31}{96}\sigma^{12} \\ A_{-1} &= \tfrac{19}{16}\sigma^6 - \tfrac{7}{8}\sigma^9 - \tfrac{157}{288}\sigma^{12} \\ A_2 &= \tfrac{25}{256}\sigma^{12}; \end{aligned} \tag{16}$$

all other coefficients start with higher powers of the parameter.

The Jacobi Constant in Terms of the Period

Hill's differential equations admit the first integral

$$\dot{u}\dot{v} - \frac{3}{4}(u+v)^2 - \frac{2}{(uv)^{1/2}} = -J, \tag{17}$$

where J is the so-called Jacobi constant.

In terms of the Cartesian coordinates already introduced this expression becomes:

$$3x^2 + \frac{2\mu}{(x^2+y^2)^{1/2}} = w^2 + J\mu^{2/3}, \tag{17'}$$

where w is the magnitude of the volocity vector of P_3 with respect to P_1 in the rotating frame. For each posi-

tion along a fixed trajectory of the Lunar Theory the velocity and position of the celestial body P_3 will satisfy the above relationship with the same value of the constant.

For a periodic trajectory the Jacobi integral will define J as a function of the period $\tau = 2\pi\sigma^3$. The knowledge of $J(\sigma^3)$ is of considerable interest in the study of periodic solutions and in a certain way will complement the equations of the orbit furnished by (15).

In order to get the expression of J in terms of σ^3 we replace the (13) and their derivatives into (17), eliminate the term with fractional exponent by means of one of Hill's equations and collect the constant terms. After some simplifications and having chosen the upper signs (corresponding to direct motion), we obtain the infinite series:

$$J = \frac{1}{\sigma^2}\sum_{-\infty}^{\infty}{}_k\left\{\left[(2k+1)^2 + 4(2k+1)\sigma^3 + \frac{9}{2}\sigma^6\right]A_k^2 + \frac{9}{2}\sigma^6 A_k A_{-k-1}\right\}. \tag{18}$$

An approximate expression up to the fourth power of the parameter σ^3 can be easily obtained by using (16):

$$J = \frac{1}{\sigma^2}\left(1 + \frac{8}{3}\sigma^3 + \frac{7}{18}\sigma^6 - \frac{140}{81}\sigma^9 - \frac{39{,}533}{7{,}776}\sigma^{12}\right). \tag{19}$$

The Jacobi integral (17′) with $w = 0$ defines the system of "zero-velocity" curves which can be used to ascertain the regions of space where the motion of P_3 can physically occur.

The singular points of such a system are at

$$y = 0, \qquad x = \pm\left(\frac{\mu}{3}\right)^{1/3},$$

and correspond to the value $J_0 = (3)^{4/3}$ of the Jacobi constant.

For $J > J_0$ the curves of zero velocity consist of two components, one of which is a closed curve Γ surrounding P_1.

The interior of Γ is a region of stability for the motion of P_3, meaning by that, that the third body, once within it, can never leave the neighborhood of the planet thus ensuring a permanent satellite.

For small values of σ^3 the Jacobi constant will take values larger than J_0 and it is easy to ascertain that the periodic orbits lie within the region of stability.

Numerical Results

It is interesting to have numerical results from the series expansions already introduced.

It is even possible to examine general characters of the configuration of the periodic orbits and the fundamental properties of the motion of a satellite along them.

For this purpose, it is sufficient to consider second-order approximations for some elements like velocity w, radius vector R = distance (P_1P_3), angular displacement θ, measured from the x-axis of the rotating system and angular velocity $\omega = \dot{\theta}$.

The velocity w can be obtained from the Jacobi integral and $R = \mu^{1/3}(uv)^{1/2}$ from (13): both elements turn out to be infinite series of

$$\cos\left(4\pi l\frac{t}{\tau}\right) \quad \text{for} \quad l = 0, 1, 2, \cdots$$

where $\tau = 2\pi\sigma^3$ is the period of the orbit.

Up to the second power of σ^3 we have:

$$w = \frac{1}{\sigma}\mu^{1/3}\left[\left(1 - \frac{2}{3}\sigma^3 + \frac{7}{18}\sigma^6\right) + \frac{7}{4}\sigma^6\cos\left(4\pi\frac{t}{\tau}\right)\right] \tag{20}$$

$$R = (x^2 + y^2)^{1/2}$$

$$= \mu^{1/3}\sigma^2\left[\left(1 - \frac{2}{3}\sigma^3 + \frac{7}{18}\sigma^6\right) - \sigma^6\cos\left(4\pi\frac{t}{\tau}\right)\right].$$

θ can be obtained from $u/v = \exp(2i\theta)$ and (13) by taking the natural logarithm and considering a series expansion, the result is an infinite series of

$$\sin\left(4\pi l\frac{t}{\tau}\right), \qquad (l = 0, 1, 2, \cdots),$$

approximated by

$$\theta = 2\pi\frac{t}{\tau} + \frac{11}{8}\sigma^6\sin\left(4\pi\frac{t}{\tau}\right), \tag{20a}$$

therefore,

$$\omega = \dot{\theta} = \frac{2\pi}{\tau} + \frac{11}{4}\sigma^3\cos\left(4\pi\frac{t}{\tau}\right). \tag{20′}$$

The above formulae are almost self-explanatory. The last one gives us the deviation from uniform motion, whereas the first two tell us that the velocity is maximum on the x-axis and minimum on the y-axis; the radius vector reaches a maximum on the y-axis and a minimum on the x-axis. These periodic trajectories look therefore, like ovals elongated in the direction perpendicular to the Sun-Planet line.

Numerical cases have been worked out for a satellite of Venus. In such case, the unit of length is the distance $R_{12} = 67{,}270.10^3$ statute miles; the unit of velocity is $\omega^* R_{12} = 22$ statute miles/second; the sidereal period of Venus has been taken as $\tau^* = 224.7008$ mean solar days; and the mass ratio as $\mu = 1/408{,}001$.

Evidently, the numerical value of σ^3, appearing in the formulae, expresses the period of the satellite around Venus as a fraction of the sidereal period of Venus about the Sun.

For the purpose of accurate numerical results, a program has been run on the IBM 704. The program essentially consists in determining the coefficients $A_{j;k}$

TABLE 1

$\sigma^3 = 0.08$			$J = 6.543651$		
t/τ	x	y	R	w	θ (degrees)
0.000	0.235928(−2)	0.000000	0.235928(−2)	0.300848(−1)	0.0000
0.025	0.232989(−2)	0.376450(−3)	0.236011(−2)	0.300658(−1)	9.1799
0.050	0.224252(−2)	0.743316(−3)	0.236250(−2)	0.300109(−1)	18.342
0.075	0.209953(−2)	0.109132(−2)	0.236622(−2)	0.299254(−1)	27.470
0.100	0.190474(−2)	0.141178(−2)	0.237090(−2)	0.298178(−1)	36.552
0.125	0.166327(−2)	0.169682(−2)	0.237606(−2)	0.296989(−1)	45.581
0.150	0.138133(−2)	0.193961(−2)	0.238121(−2)	0.295802(−1)	54.553
0.175	0.106602(−2)	0.213444(−2)	0.238584(−2)	0.294734(−1)	63.473
0.200	0.725128(−3)	0.227682(−2)	0.238950(−2)	0.293887(−1)	72.348
0.225	0.366925(−3)	0.236354(−2)	0.239185(−2)	0.293345(−1)	81.191
0.250	0.000000	0.239266(−2)	0.239266(−2)	0.293159(−1)	90.000

Elements of the orbit for a satellite of Venus moving according to direct motion with period $\tau = 17.9761$ days; x, y are synodical coordinates with center at Venus; R is the radius vector; w is the velocity; θ the angular displacement measured from the Sun-Venus line; J is the Jacobi constant. Unit of length = 67,270.10³ statute miles; unit of velocity = 22 statute miles/second. The numbers within parentheses denote powers of ten.

from the recurrent procedure mentioned in the first part of the paper, and which is the key to any rapid numerical evaluation. This has been done for values of j from one to twelve and the corresponding values of k.

Some of the coefficients (which we do not reproduce here for lack of space) were found to be zero, so that while the series were summed up to the twelfth power of σ^3, the maximum value of $|k|$ appearing in the set of coefficients was six. The values of x, y, R, θ, w, and J were printed as functions of σ^3 and for values of t/τ between zero and 0.25. Good results have been obtained up to $\sigma^3 = 0.4$.

Table 1 is a brief sketch of numerical data for a satellite having a direct periodic motion around Venus. σ^3 was chosen to be 0.08 which means that the period of the satellite is 17.9761 days. The unit of length is the mean distance of Venus to the Sun and the unit of velocity is the orbital velocity of Venus around the Sun, supposed constant; θ is measured in degrees starting from the x-axis, the numbers within parentheses represent powers of ten.

Error Analysis

An error analysis for periodic trajectories seems to be essential from a practical point of view. The conditions which give rise to periodic orbits have been examined in the first part of this paper. However, it is to be expected that trajectories which lie in a neighborhood of a periodic one might be of interest for the practical problem of tracking or forecasting the position of a satellite which fails to be injected into a periodic orbit because of engine malfunction or other causes.

In general, these trajectories are not periodic and take into consideration the perturbation effects of higher order.

The general theory of trajectories, lying in a neighborhood of a fixed one, can be found in [8] pages 65, 75, 170–173, and [4] pages 335–344. Let $x(t)$, $y(t)$ be the coordinates of a periodic trajectory and $x + \Delta x$, $y + \Delta y$ the coordinates of a neighboring trajectory corresponding to the same value of the total energy as along the periodic orbit. The functions Δx, Δy must then satisfy a linear system of differential equations (the so-called equations of variation or Jacobi equations) and can be visualized as isoenergetic displacements of the coordinates of the periodic trajectory.

For every position of $P_3(x,\ y)$ moving along a periodic orbit, let us consider the intrinsic orthogonal reference formed by the tangent line to the orbit, positive in the direction of motion and the normal line, positive toward the interior of the trajectory. The position of $\bar{P}_3(x + \Delta x; y + \Delta y)$ moving along a neighboring trajectory and corresponding to the same value of time as P_3 can be fixed by means of its coordinates $(p; q)$ with respect to the above reference.

With the notation already used, we have

$$\begin{aligned} \Delta x &= p\,\frac{\dot{x}}{w} - q\,\frac{\dot{y}}{w} \\ \Delta y &= p\,\frac{\dot{y}}{w} + q\,\frac{\dot{x}}{w}, \end{aligned} \tag{21}$$

where the velocity w is obtainable from the Jacobi integral:

$$\begin{aligned} w^2 &= 2[U(x;y) - J\mu^{2/3}] \\ U(x;y) &\equiv \frac{3}{2}x^2 + \frac{\mu}{R}. \end{aligned} \tag{22}$$

It is easy to show that the isoenergetic normal displacement q satisfies the equation

$$\ddot{q} + K(t)q = 0, \tag{23}$$

where

$$K(t) \equiv \frac{\ddot{w}}{w} + 2\left(1 + \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{w^2}\right)^2 + 2 - (U_{xx} + U_{yy}); \tag{23a}$$

whereas the tangential displacement p is obtainable by numerical integration from

$$p = 2w \int \frac{q}{w}\left(1 + \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{w^2}\right) dt. \tag{24}$$

In terms of the complex conjugate variables (u, v) we can write from (23a) and (2)

$$\begin{aligned} K(t) \equiv 2\left[1 + \frac{1}{2i}\left(\frac{\ddot{u}}{\dot{u}} - \frac{\ddot{v}}{\dot{v}}\right)\right]^2 + \frac{1}{2}\left(\frac{\ddot{u}}{\dot{u}} + \frac{\ddot{v}}{\dot{v}}\right)^{\cdot} \\ + \frac{1}{4}\left(\frac{\ddot{u}}{\dot{u}} + \frac{\ddot{v}}{\dot{v}}\right)^2 - \left[1 + \frac{1}{(uv)^{3/2}}\right]. \end{aligned} \tag{23'}$$

This expression is suitable for finding a series expression of $K(t)$. Using (7′), (7), and (4) the final results can be put in the following form:

$$K(t) \equiv \frac{1}{\sigma^6}\left[K_0 - 2\sum_1^\infty {}_l K_l \cos\left(\frac{2lt}{\sigma^3}\right)\right], \tag{25}$$

where K_l $(l = 0, 1, 2, \cdots)$ are infinite series in σ^3, starting with terms in σ^{6l}.

Carrying out the calculations for the first two coefficients it was found

$$\begin{aligned} K_0 &\equiv 1 + 2\sigma^3 - \tfrac{1}{2}\sigma^6 + [12] \\ K_1 &\equiv \tfrac{15}{2}\sigma^6 + \tfrac{57}{4}\sigma^9 + [12], \end{aligned} \tag{26}$$

where the numbers within brackets denote powers of σ. Neglecting the fourth power of the parameter σ^3 we realize that equation (23) is of the Mathieu type. For sake of convenience, we let $T = t/\sigma^3$ so that (23) becomes:

$$q'' + (K_0 - 2K_1 \cos 2T)\, q = 0, \tag{27}$$

where the primes denote derivatives with respect to T. For small values of σ^3 the coefficients K_0, K_1 will be both positive and, in general, not integers. It is to be understood that in order to have a complete picture of the situation in the neighborhood of a periodic trajectory, the general solution of (27) is needed.

Various methods for solving Mathieu equations are available (see [7] page 404 and [2] chapter 5). The procedure followed in this paper is very convenient for numerical calculations when the coefficient K_1 takes on small values.

The general solution of (27), depending upon two arbitrary parameters, is

$$q(T; Q_0; \beta) \equiv Q_0 \sum_{-\infty}^{\infty} {}_k \frac{Q_k}{Q_0} \cos[(2k + c)T - \beta], \tag{28}$$

where the coefficients Q_k are infinite power series in K_1. The solution will be periodic only if c is a rational number.

Since K_1 is a small quantity, it is convenient to express the general solution as an infinite power series in K_1. This fact will be used to find the coefficients of (28).

Let us remark that when $K_1 = 0$, the general solution of (27) is proportional to $\cos(\sqrt{K_0}T - \beta)$. Consequently when $K_1 \neq 0$ but takes on small values (this is actually the situation which arises), we can write

$$\begin{aligned} \bar{q}(T) &\equiv q(T; 1; 0) \equiv \cos(cT) + \sum_1^\infty {}_k K_1^k E_k(T) \\ K_0 &= c^2 + \sum_1^\infty {}_k K_1^k F_k. \end{aligned} \tag{29}$$

Here the F_k's are undetermined constants and the E_k's are functions to be determined which must necessarily be combinations of terms like $\cos[(2k + c)T]$.

Replacing these trial series into the Mathieu equation and equating to zero the coefficients of the various powers of q, we have:

$$\begin{aligned} E_1'' + c^2E_1 &= \cos[(c-2)T] + \cos[(c+2)T] - F_1\cos(cT) \\ E_k'' + c^2E_k &= 2E_{k-1}\cos(2T) - F_k\cos(cT) \\ &\quad - (E_1F_{k-1} + \cdots + E_{k-1}F_1) \\ &\qquad (k = 2, 3, \cdots). \end{aligned} \tag{30}$$

The above written expressions are linear differential equations with constant coefficients which will allow us to determine the unknown functions E_k by means of a recurrent procedure. It is sufficient to find particular solutions of such equations; furthermore, it is easy to realize that the F_k's can be determined from the property that the solutions cannot contain terms like $T\cos(cT)$. If c is not an integer, the above procedure furnishes the following result (up to the third power of K_1):

$$\begin{aligned} \bar{q}(T) &= \cos(cT) - \frac{1}{4}K_1\left[\frac{\cos(c+2)T}{c+1} - \frac{\cos(c-2)T}{c-1}\right] + \frac{1}{32}K_1^2\left[\frac{\cos(c+4)T}{(c+1)(c+2)} + \frac{\cos(c-4)T}{(c-1)(c-2)}\right] \\ &\quad - \frac{1}{128}K_1^3\left[\frac{(c^2+4c+7)\cos(c+2)T}{(c-1)(c+1)^3(c+2)} - \frac{(c^2-4c+7)\cos(c-2)T}{(c+1)(c-1)^3(c-2)} \right. \\ &\quad \left. + \frac{\cos(c+6)T}{3(c+1)(c+2)(c+3)} - \frac{\cos(c-6)T}{3(c-1)(c-2)(c-3)}\right] + \cdots; \\ K_0 &= c^2 + \frac{K_1^2}{2(c^2-1)} + \frac{(5c^2+7)K_1^4}{32(c^2-1)^3(c^2-4)} + \cdots. \end{aligned} \tag{31}$$

The second relationship, written above, should be used to determine the value of the parameter c; however, it is not very convenient for numerical calculations since a high order algebraic equation in c should be solved in order to get a good approximation. We shall follow a different procedure to obtain the value of c explicitly in terms of K_0 and K_1. From the general theory of Mathieu equations, it is known that if

$q^*(T; K_0, K_1)$ is the particular solution of (27), such that

$$q^*(0; K_0, K_1) = 1 \quad \text{and} \quad q^{*\prime}(0; K_0, K_1) = 0,$$

then

$$\cos(\pi c) = q^*(\pi; K_0, K_1), \tag{32}$$

see [1] page 208 and [3] page 103.

Expanding q^* into a power series of K_1

$$q^*(T; K_0, K_1) \equiv \sum_0^\infty {}_r K_1^r q_r(T; K_0), \tag{33}$$

where

$$q_r(T; K_0) \equiv \frac{1}{r!}\left[\frac{\partial^r q^*(T; K_0, K_1)}{\partial K_1^r}\right]_{K_1=0},$$

and replacing such expansion (33) into (27) we realize that

$$q_0(T; K_0) = \cos\sqrt{K_0}T, \tag{34}$$

whereas the other q_r's satisfy the recurrent system of linear differential equations

$$q_r''(T; K_0) + K_0 q_r(T; K_0) = 2\cos(2T) q_{r-1}(T; K_0) \qquad (r = 1, 2, 3, \cdots), \tag{35}$$

with initial conditions

$$q_r(0; K_0) \doteq q_r'(0; K_0) = 0. \tag{35'}$$

Hence, up to the fourth power of K_1 it was obtained from (32), (34), and (35):

$$\cos(\pi c) = \cos(\pi\sqrt{K_0}) + K_1^2 \frac{\pi \sin(\pi\sqrt{K_0})}{4\sqrt{K_0}(K_0 - 1)} + K_1^4\left[\frac{(15K_0^2 - 32K_0 + 8)\pi \sin(\pi\sqrt{K_0})}{64(K_0 - 1)^3(K_0 - 4)K_0\sqrt{K_0}} - \frac{\pi^2 \cos(\pi\sqrt{K_0})}{32K_0(K_0 - 1)^2}\right]. \tag{36}$$

This formula gives explicitly the value of $\cos(\pi c)$. The values of c thus obtained differ by an arbitrary additive constant $\pm 2k$ (where k is an integer); the value to be chosen must be the closest to $\sqrt{K_0}$. In general, c is not an integer. If this is not the case, the solution of the Mathieu equation (27) can be obtained from standard procedures available in many textbooks, see e.g., [2] pages 15–17 and [7] page 404.

The results expressed by formulae (31) and (36) complete the required elements for the general solution (28) of (27), the two parameters Q_0 and β being determined from the initial conditions (i.e., position and velocity). The knowledge of $q(t/\sigma^3)$ and of the elements of a periodic orbit (x, y, w and their derivatives) allow us to obtain the tangential displacement p from (24) by means of a numerical quadrature. Equations (21) finally yield Δx and Δy in terms of the time.

Equation (23) is the well-known Hill equation and turns out to be of fundamental importance in the problem of the stability of periodic orbits, a problem which is strictly connected with the subject matter of this section. Among the numerous literature on this topic one should mention a recent paper by P. J. Message "Some periodic orbits in the restricted problem of three bodies and their stabilities" published in the Astronomical Journal **64,** 226, 1959.

Conclusions

The trajectory of a stable satellite around a planet, when the Sun's perturbation is taken into consideration, should be a periodic one.

The periodic trajectories obtained in this paper from the point of view of the Lunar Theory, are symmetrical with respect to the axes of a synodic reference with the origin at the planet; they are oval-shaped with the planet at the center of symmetry and slightly elongated in the direction perpendicular to the Sun-Planet line. The coordinates of such trajectories have been expressed as cosine and sine series of time. A recurrent procedure allows us to obtain the coefficients of the expansions, once the period of revolution has been chosen, with any required degree of accuracy; the procedure is furthermore fit for programming on a high-speed electronic computer. The Jacobi constant of the motion can likewise be computed in terms of the period, so that from the Jacobi integral it is possible to obtain the magnitude of the velocity for positions along an orbit of such period. The numerical value of the Jacobi constant is therefore indispensable for evaluating the requirements of a retrorocket so that a ballistic missile can be injected into a periodic orbit.

The numerical work undertaken for a satellite of Venus yields good results for trajectories whose period is up to 0.4 of the sidereal period of Venus. To account for any practical situation, deviations from a periodic orbit have been studied as displacements of the positions of the periodic orbit itself. The numerical methods for such calculations expressed in the paper seem quite straightforward for applications.

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On Strömgren's Method of Special Perturbations

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Abstract

In this paper Strömgren's method for the numerical computation of perturbations is modified to include the effects of higher order. The accurate form for the rotation matrix in terms of, so called, Gibbs vector is given and the differential equation for the perturbations of this vector is established. A similar modification is applied also to Hansen's method.

Introduction

In this paper Strömgren's method (1929) for computation of special perturbations is modified to include the effects of higher order. The standard form of Strömgren's method possesses a mathematical elegance and simplicity and it gives the vectorial elements of the orbit without trigonometrical transformations. The disadvantage of this method is that it takes into account perturbations of the first order only. The integrated value of the angular velocity of rotation of the osculating ellipse is used by Strömgren to obtain the Gibbs rotation vector, and because this vector is not obtained directly, only the first approximation to the matrix of rotation can be easily deduced.

The author decided to reconsider the possibility of returning to Strömgren's method and to deduce the accurate equations for the Gibbs vector. Using this vector a closed form for the rotation matrix is also obtained.

Some methods developed by S. Herrick (1948) and by the author (1954) avoid the forming of the matrix of rotation by using three unit vectors rigidly connected with the osculating ellipse, but, in these methods the eccentricity appears as a "small divisor" in more equations than in Strömgren's method. This fact limits the applicability of all three methods to case of moderate or large eccentricities. The new form of Herrick's method (1953, 1961) avoids the division by e completely and can be used also for the cases of small eccentricities.

The division by e does not appear also in Hansen's method. The form of Hansen's method which is convenient for the computation of special perturbations for all eccentricities is given in the last part of this article.

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The Basic Equations

Let **R** be the unit vector normal to the osculati orbit plane, **P** be the unit vector directed from th central body to the osculating perihelion, and **Q** **R** × **P**. The values of these vectors for the initi moment will be designated by $\mathbf{P}_0$, $\mathbf{Q}_0$, $\mathbf{R}_0$. As in t author's previous article (1954) the equation of t motion is taken in the form

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3} + \mathbf{F},$$

and the projections of the disturbing force **F** on **Q**, **R**, **r** and **R** × **r** are designated by L, N, W, S and respectively. Let ϑ, φ and ψ be the inclination of t orbit, the longitude of the ascending node and t argument of perigee of the osculating orbit, where t system $\mathbf{P}_0$, $\mathbf{Q}_0$, $\mathbf{R}_0$ is taken as a reference system. P

$$g_1 = \frac{\cos\frac{1}{2}(\varphi - \psi)}{\cos\frac{1}{2}(\varphi + \psi)}\, tg\,\frac{1}{2}\vartheta$$

$$g_2 = \frac{\sin\frac{1}{2}(\varphi - \psi)}{\cos\frac{1}{2}(\varphi + \psi)}\, tg\,\frac{1}{2}\vartheta$$

$$g_3 = tg\,\tfrac{1}{2}(\varphi + \psi)$$

$$\mathbf{g} = g_1\mathbf{P}_0 + g_2\mathbf{Q}_0 + g_3\mathbf{R}_0 .$$

The matrix of rotation Γ, transforming the syste ($\mathbf{P}_0$, $\mathbf{Q}_0$, $\mathbf{R}_0$) into the system (**P**, **Q**, **R**), can be wri ten in the forms suggested by Gibbs (1901)

$$\Gamma = I + \frac{2}{1 + g^2}[\mathbf{g} \times I + \mathbf{g} \times (\mathbf{g} \times I)],$$

or

$$\Gamma = \frac{1 - g^2}{1 + g^2} I + \frac{2\mathbf{g} \times I + 2\mathbf{gg}}{1 + g^2},$$

where I is the unit matrix,

$$I = \mathbf{P}_0\mathbf{P}_0 + \mathbf{Q}_0\mathbf{Q}_0 + \mathbf{R}_0\mathbf{R}_0 .$$

The equation (6) or (6′) replaces the development Γ into an infinite series (Herget, 1948). Let **g** and be two Gibbs vectors. If the rotation defined by $\boldsymbol{\gamma}$ fo lows the rotation defined by **g**, then Gibbs vector the combined rotation is given by the formula (Gibb 1901).

$$\mathbf{G} = \frac{\mathbf{g} + \boldsymbol{\gamma} + \boldsymbol{\gamma} \times \mathbf{g}}{1 - \mathbf{g}\cdot\boldsymbol{\gamma}}.$$

The osculating unit vectors **P**, **Q**, **R** are given by the equation of the form:

$$\begin{aligned} \mathbf{U} &= \mathbf{U}_0 + \frac{2}{1+g^2}[\mathbf{g}\times\mathbf{U}_0 + \mathbf{g}\times(\mathbf{g}\times\mathbf{U}_0)] \\ &= \frac{1-g^2}{1+g^2}\mathbf{U}_0 + \frac{2\mathbf{g}\times\mathbf{U}_0 + 2\mathbf{gg}\cdot\mathbf{U}_0}{1+g^2}, \end{aligned} \tag{7}$$

where **P**, **Q** or **R** must be substituted for **U** and $\mathbf{P}_0$, $\mathbf{Q}_0$, $\mathbf{R}_0$ for $\mathbf{U}_0$. After **P**, **Q** and **R** are obtained the osculating position vector is computed using the standard formula:

$$\mathbf{r} = \mathbf{P}a(\cos E - e) + \mathbf{Q}a\sqrt{1-e^2}\sin E. \tag{8}$$

Let $\mathbf{\Omega}$ be the instantaneous angular velocity of rotation of the osculating ellipse and ω_1, ω_2, ω_3 the projections of $\mathbf{\Omega}$ in the directions of **P**, **Q**, **R**, we have (Strömgren, 1929):

$$\omega_1 = hWa(\cos E - e) \tag{9}$$

$$\omega_2 = hWa\sqrt{1-e^2}\sin E \tag{10}$$

$$\omega_3 = \frac{\sqrt{a}}{e}(T\sin E - \sqrt{1-e^2}\,L) \tag{11}$$

$$h = \frac{1}{\sqrt{a(1-e^2)}}.$$

Let **m** be the unit vector along the line of nodes in the system ($\mathbf{P}_0$, $\mathbf{Q}_0$, $\mathbf{R}_0$). It is kinematically evident that

$$\begin{aligned} \mathbf{\Omega} &= \omega_1\mathbf{P} + \omega_2\mathbf{Q} + \omega_3\mathbf{R} \\ &= \frac{d\vartheta}{dt}\mathbf{m} + \frac{1}{2}(\mathbf{R}_0 + \mathbf{R})\frac{d(\varphi+\psi)}{dt} \\ &\qquad + \frac{1}{2}(\mathbf{R}_0 - \mathbf{R})\frac{d(\varphi-\psi)}{dt}. \end{aligned} \tag{12}$$

Put

$$\boldsymbol{\omega} = \omega_1\mathbf{P}_0 + \omega_2\mathbf{Q}_0 + \omega_3\mathbf{R}_0. \tag{13}$$

We have

$$\mathbf{\Omega} = \Gamma\cdot\boldsymbol{\omega} = \boldsymbol{\omega} + \frac{2}{1+g^2}[\mathbf{g}\times\boldsymbol{\omega} + \mathbf{g}\times(\mathbf{g}\times\boldsymbol{\omega})]. \tag{14}$$

Let **g** and **g** + d**g** be two Gibbs vectors defining the rotations of the osculating ellipse from the moment t_0 to the moments t and $t + dt$ respectively. The infinitesimal matrix of rotation which brings the osculating ellipse from the position at the moment t to the position at the moment $t + dt$ is

$$I + \boldsymbol{\omega}\,dt \times I,$$

and the corresponding Gibbs vector is $\frac{1}{2}\boldsymbol{\omega}\,dt$. Substituting

$$\mathbf{G} = \mathbf{g} + d\mathbf{g}, \qquad \mathbf{g} = \mathbf{g}, \qquad \gamma = \tfrac{1}{2}\boldsymbol{\omega}\,dt,$$

into (6″), we deduce

$$\frac{d\mathbf{g}}{dt} = \frac{1}{2}\boldsymbol{\omega} + \frac{1}{2}\mathbf{g}\times\boldsymbol{\omega} + \frac{1}{2}\mathbf{gg}\cdot\boldsymbol{\omega}, \tag{15}$$

or

$$\frac{d\mathbf{g}}{dt} = \frac{1}{2}(I + \mathbf{g}\times I + \mathbf{gg})\cdot\boldsymbol{\omega}. \tag{16}$$

We find from (6′)

$$\tfrac{1}{2}(1+g^2)(\Gamma + I) = I + \mathbf{g}\times I + \mathbf{gg}. \tag{17}$$

From (17), (16) and (14) we deduce

$$\frac{d\mathbf{g}}{dt} = \frac{1}{4}(1+g^2)(\mathbf{\Omega} + \boldsymbol{\omega}), \tag{18}$$

and also

$$\frac{d\mathbf{g}}{dt} = \frac{1}{2}(1+g^2)\boldsymbol{\omega} + \frac{1}{2}\mathbf{g}\times\boldsymbol{\omega} + \frac{1}{2}\mathbf{g}\times(\mathbf{g}\times\boldsymbol{\omega}). \tag{19}$$

Any one of the equations (15), (18) or (19) gives Gibbs vector **g** accurately and replaces Strömgren's equations for $d\theta_x/dt$, $d\theta_y/dt$, $ed\pi/dt$. If the second and the third terms on the right hand side of (15) are neglected, the result coincides with the basic Strömgren's equation. Multiplying (16) by the matrix

$$\frac{2}{1+g^2}(I - \mathbf{g}\times I),$$

and considering the following formulae,

$$I\cdot\mathbf{A} = \mathbf{A}$$

$$\mathbf{A}\times I\cdot\mathbf{B} = \mathbf{A}\times\mathbf{B},$$

$$(\mathbf{g}\times I)\cdot(\mathbf{g}\times I) = \mathbf{gg} - \mathbf{g}^2 I$$

we deduce

$$\boldsymbol{\omega} = \frac{2}{1+g^2}(I - \mathbf{g}\times I)\cdot\frac{d\mathbf{g}}{dt},$$

or

$$\boldsymbol{\omega} = \frac{2}{1+g^2}\left(\frac{d\mathbf{g}}{dt} + \frac{d\mathbf{g}}{dt}\times\mathbf{g}\right). \tag{20}$$

This vectorial formula is identical with the classic scalar formulae (Whittaker, 1904):

$$\omega_1 = 2\left(+\chi\frac{d\xi}{dt} + \zeta\frac{d\eta}{dt} - \eta\frac{d\zeta}{dt} - \xi\frac{d\chi}{dt}\right)$$

$$\omega_2 = 2\left(-\zeta\frac{d\xi}{dt} + \chi\frac{d\eta}{dt} + \xi\frac{d\zeta}{dt} - \eta\frac{d\chi}{dt}\right)$$

$$\omega_3 = 2\left(+\eta\frac{d\xi}{dt} - \xi\frac{d\eta}{dt} + \chi\frac{d\zeta}{dt} - \zeta\frac{d\chi}{dt}\right),$$

where ξ, η, ζ, χ are Euler's parameters

$$\xi = \sin\frac{1}{2}\vartheta\cos\frac{1}{2}(\varphi-\psi) = \frac{g_1}{\sqrt{1+g^2}}$$

$$\eta = \sin\frac{1}{2}\vartheta\sin\frac{1}{2}(\varphi-\psi) = \frac{g_2}{\sqrt{1+g^2}}$$

$$\zeta = \cos\frac{1}{2}\vartheta\sin\frac{1}{2}(\varphi+\psi) = \frac{g_3}{\sqrt{1+g^2}}$$

$$\chi = \cos\frac{1}{2}\vartheta\cos\frac{1}{2}(\varphi+\psi) = \frac{1}{\sqrt{1+g^2}}.$$

We have for the perturbations of the mean anomaly (Brown, 1896)

$$\frac{d\Delta M}{dt} = -\frac{2rS}{\sqrt{a}} - \sqrt{1-e^2}\,\omega_3. \tag{21}$$

Multiplying (12) by $\mathbf{R}_0 + \mathbf{R}$, we deduce

$$\frac{d(\varphi+\psi)}{dt} = \omega_3 + \frac{(\omega_1\mathbf{P} + \omega_2\mathbf{Q})\cdot\mathbf{R}_0(1+g^2)}{2(1+g_3^2)}, \tag{22}$$

or, taking (7) into consideration,

$$\frac{d(\varphi+\psi)}{dt} = \omega_3 + \frac{\omega_1(g_1g_3 - g_2) + \omega_2(g_2g_3 + g_1)}{1+g_3^2}. \tag{23}$$

Adding (21) and (23) together we obtain an equation for the mean longitude L_1 in the system $(\mathbf{P}_0, \mathbf{Q}_0, \mathbf{R}_0)$,

$$\frac{dL_1}{dt} = -\frac{2rS}{\sqrt{a}} + \frac{e}{1+\sqrt{1-e^2}}(e\omega_3) + \frac{\omega_1(g_1g_3 - g_2) + \omega_2(g_2g_3 + g_1)}{1+g_3^2}. \tag{24}$$

This last equation replaces Strömgren's equation for dL_1/dt. The equations

$$\frac{de}{dt} = \sqrt{a(1-e^2)}\,(T\cos E + N) \tag{25}$$

$$\frac{dn}{dt} = -\frac{3}{a\sqrt{1-e^2}}(T + eN), \tag{26}$$

remain unchanged. Kepler's equation takes the form

$$E - e\sin E = M_0 + (\Delta L_1 - 2\,arctg\,g_3) + n_0(t - t_0) + \iint \frac{dn}{dt}\,dt^2. \tag{27}$$

The values (9) — (11) of ω_1, ω_2, ω_3 must be used in the equations (15) — (24). The equations (15), or (18), or (19) and (24), (25), (26) and (27) constitute the basic system to be programmed for the use of electronic machine. In this modified Strömgren's method the danger of losing the accuracy because of the small divisor is not greater than in the classic Lagrange method and the matrix of rotation is formed directly, using three independent parameters. The developed equations can be easily programmed and the author's opinion is, that it is desirable that Strömgren's method come into wider use again.

A similar system of formulae can also be developed for the use of Hansen's method (1857) to compute the special perturbations. Let **P**, **Q**, **R** now be the system rigidly connected with the osculating orbit plane (not with the osculating ellipse as in Strömgren's method). The basic Hansen's ellipse has the fixed value a_0 of the semi-major axis and of the eccentricity e_0 and it does not move relative to the system $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$. Let $1+\nu$ and ΔM be Hansen's perturbations in the radius-vector and the mean anomaly. The position vector $\mathbf{r}$ of the disturbed body is given by the equations

$$\mathbf{r} = (1+\nu)\boldsymbol{\varrho} \tag{28}$$

$$\boldsymbol{\varrho} = \mathbf{P}a_0(\cos E - e_0) + \mathbf{Q}a_0\sqrt{1-e_0^2}\sin E \tag{29}$$

$$E - e_0\sin E = M_0 + n_0(t-t_0) + \Delta M. \tag{30}$$

The instantaneous angular velocity of rotation of the system $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ is

$$\boldsymbol{\Omega} = (1+\nu)hW\boldsymbol{\varrho}. \tag{31}$$

Put

$$\boldsymbol{\omega} = (1+\nu)hW\boldsymbol{\varrho}_0, \tag{32}$$

$$\boldsymbol{\varrho}_0 = \mathbf{P}_0a_0(\cos E - e_0) + \mathbf{Q}_0a_0\sqrt{1-e_0^2}\sin E. \tag{33}$$

Substituting the values of $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ into (18) we have

$$\frac{d\mathbf{g}}{dt} = \frac{1+g^2}{4}hW(\boldsymbol{\varrho} + \boldsymbol{\varrho}_0)(1+\nu) \tag{34}$$

and evidently,

$$\boldsymbol{\varrho} = \boldsymbol{\varrho}_0 + \frac{2}{1+g^2}[\mathbf{g}\times\boldsymbol{\varrho}_0 + \mathbf{g}\times(\mathbf{g}\times\boldsymbol{\varrho}_0)]. \tag{35}$$

The remaining system of formulae can be retained in the form given in the author's previous article (1958):

$$\overline{w} = X + Y(\cos E - e_0) + Z\sqrt{1-e_0^2}\sin E \tag{36}$$

$$\Delta = \frac{h_0}{h} - 1, \qquad h_0 = \frac{1}{\sqrt{a_0(1-e_0^2)}} \tag{37}$$

$$\frac{d\Delta}{dt} = h_0(1+\nu)\rho T \tag{38}$$

$$\frac{dY}{dt} = 2h_0\cdot a_0\left[\frac{h^2}{h_0^2}(1+\nu)\cdot T\cos E + N\right] \tag{39}$$

$$\frac{dZ}{dt} = 2h_0\cdot a_0\left[\frac{h^2}{h_0^2}\cdot\frac{1+\nu}{\sqrt{1-e_0^2}}T\sin E - L\right] \tag{40}$$

$$X = -\frac{(3+\Delta)\Delta}{1+\Delta} \tag{41}$$

$$\frac{1}{1+\nu} = 1 + \frac{\overline{w} - \Delta}{2(1+\Delta)} \tag{42}$$

and (Hill, 1881)

$$\frac{d\Delta M}{dt} = n_0\frac{\overline{w} + \nu^2}{1-\nu^2}. \tag{43}$$

The system of formulae (34) — (43) does not contain the eccentricity as a divisor and, consequently, it can be used for all orbits. This system was programmed at the Minor Planets Center in Cincinnati for the computation of special perturbations of celestial bodies.

Acknowledgement

The author takes this opportunity to express his gratitude to his colleague Mrs. Ann Bailie for careful checking of the manuscript.

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Goddard and His Early Rockets: 1882—1930

E. R. Hagemann

bstract

This paper traces the career of Dr. Robert Hutchings ;oddard from his earliest rocket experiments as an under-;raduate at Worcester Polytechnic Institute to his work at Camp Devens, Massachusetts. It also attempts to demon-trate that during this period Goddard was recognized by his eers, that he was supplied with funds for research and de-elopment, and that he was a very thorough and careful cientist who eschewed the sensational in his work and find-ngs.

ntroduction

This paper is almost in the nature of an attack, an ttack on certain near legends and myths regarding the areer and achievements of Dr. Robert Hutchings ;oddard, late Professor of Physics, Clark University, Vorcester, Mass. Over the years since his death at Baltimore, Maryland, there have grown stories to the ffect that he was ignored by his peers, that he was aughed at by the knowledgeable, that he was without unds, that he labored alone and unknown, that he was prophet without honor in his own country.

Like so many legends and myths there is a grain of ruth in these stories, but for the most part they simply re not true. If this paper accomplishes nothing else, t will show that Dr. Goddard was well known in his ime; that he was amply and very often supplied with esearch grants, funds, and facilities from the Smith-onian Institution, the National Geographic Society, he Carnegie Institution, Clark University, Worcester Polytechnic Institute, and especially the Daniel and 'lorence Guggenheim Memorial Fund; that he did not abor alone; and that the public and the scientific com-nunity were almost always well aware of what he was bout (the New York *Times* never failed to publicize im, often on Page One; as a matter of fact, his widow as in her possession some nine volumes of newspaper lippings on her husband).

It is hoped, too, that this paper accomplishes some-ning in the way of destroying a pejorative tale about Goddard: that he was a "moony" man; that is, he emphasized the extravagantly romantic and somewhat unscientific aspects of rocketry, that he was a Sunday supplement writer in the vein of a present-day scientist who need not be named. Goddard, except for a minor lapse or two, was precisely the opposite. He was a very careful, very thorough scientist who although suggesting an ultimate result—a landing on the Moon—was always concerned with goals nearer at hand. He was, in his own way, a systems engineer. He was an accurate and careful mathematician. Unfortunately, the wayward press of the United States inevitably extracted the sensational from his work and stressed it.

This horrified Goddard. It stiffened him. It made him suspicious. He refrained from publishing (only three articles on rocketry were published in his lifetime). But he never slackened in his tasks and he worked steadily until his death, as he had worked all his life, on the data at hand.

Although he appears to have been a pragmatic, practical man, he was not without a romantic aura, particularly in his early life; and this places him in the tradition of the Yankee inventor, Edison, Goodyear, Firestone, Ford, et al. He was born in Worcester, Massachusetts, on October 5, 1882. Because he was a sickly and thin child, he was prohibited from the usual aimless violence of a boy's life. Instead, he read voraciously and nursed his imagination. In his middle age, Goddard recalled at the age of six having seen electricity produced by a Leclanché battery. He obtained the zinc from such a battery and scuffed along a gravel walk. He mounted a low fence and jumped. Then he repeated the experiment "scuffing over a longer distance, and endeavored to convince myself I had jumped higher."

In his sixteenth year, he says, "an event happened which was destined to provide me with all the scientific speculative material I could desire." The Boston *Post* in January 1898 serialized *Fighters from Mars*; *or*, *The War of the Worlds*, *In and Near Boston*, and followed

with Garrett P. Serviss' *Edison's Conquest of Mars*. He also read Jules Verne and was very fond of *From the Earth to the Moon* (1865) in which rockets were featured.

His teen-age mind roamed and probed without rest. In the Spring of 1898 he thought it would be perfectly delightful to fly a permanent balloon at the end of a string. He obtained 1/100-inch aluminum sheet, shaped it like a pillow, and filled it with hydrogen. It was naturally too heavy. He accepted his "failior," as he spelled it, and turned to other things: "how birds fly," for example. On October 19, 1899, he sat in a tree and "imagined how wonderful it would be to make some device which had even the *possibility* of ascending to Mars." His crude wooden models gave "negative results," and Goddard admitted "that there might be something after all to Newton's Laws, which I had read in *Cassell's Popular Educator*, given to me by my father."[1]

He was late in entering high school but not inactive there. During his first year at South High in Worcester, in 1901, he wrote an article on space navigation and submitted it unsuccessfully to *Popular Science Monthly*. Goddard writes: "The article mentioned how the idea of firing several cannons, arranged like a nest of beakers, had been proposed, and quoted figures on the frequency of meteors in space. . . ." It will be noted how he always considered various possibilities and problems. (I have mentioned his thoroughness and his carefulness.) In other words, he measured. Propulsion by firearms of some kind was a favorite idea of his, and he once considered obtaining propulsion by a machine gun device that fired bullets downward—he cannot forget Newton's third law.

Upon graduation from high school in 1904, Goddard heroically burned all his notebooks because "I knew they were erroneous." But the dream would not be forgotten and inside of two months he was again scribbling and calculating. He bought a batch of green clothbound notebooks and commenced a scientific and systematic recording of his ideas and concepts. (They are fascinating and we will return to them in a moment.)

Goddard matriculated at Worcester Polytechnic Institute and was graduated from there with the degree of Bachelor Science in 1908, holding highest honors in physics and mathematics. His academic record did not hold him in thrall: he was president of his class, editor-in-chief of the college yearbook, and a fraternity man. Nor did the textbooks hold his imagination and curiosity in thrall. In his freshman year a professor assigned "Travelling in 1950" as a theme topic. Goddard responded with a bold and brilliant idea: a railway line, with air-tight cars in a vacuum which were supported electromagnetically without metal-to-metal contact. He calculated that there would be an acceleration of 32 ft/sec^2 during the first half of the journey and a deceleration of equal amount during the last half. This meant a running time of ten minutes from Boston to New York City. Typically enough, however, Goddard was co[n]cerned with the demonstration of a principle: the co[n]tinued acceleration of a body by forces which chang[e] from attraction to repulsion as the body passed by t[he] source of the force.

It would seem that propulsion was almost an obs[es]sion with him. In his junior year at WPI he submitt[ed] seriatim, a paper to *Scientific American*, *Popular S[ci]ence Monthly*, and *Popular Astronomy* that propo[sed] heat from radioactive materials be used to expel su[b]stances at high velocity from an orifice. Jet propuls[ion] would be furnished sufficient to navigate in interpla[ne]tary space. The periodicals refused the manuscript[s].

Toward the end of his senior year, beneath a build[ing] at Worcester Tech, occurred the most significant ev[ent] of all: a smoke-filled basement, the result of static te[sts] of small rockets.

I now must return to his notebooks. They are ext[ant] and furnish us a classic example of usage by a w[ell-]grounded man. They also illustrate how a notebo[ok] might fecundate a mind.

During the Christmas recess in 1909, Goddard su[m]marized twenty-six methods in his notebooks "invo[lv]ing means in space, means taken with the apparat[us] and means sent from the Earth." There isn't the ro[om] for all of them here; however, extracts suggest the fr[uitful]ness thereof:

(1) September 6, 1906: reaction by streams of ions [to] furnish rocket propulsion. (This was sub[se]quently developed in a patent taken out May [?] 1915.)

(2) July 1907: use of solar energy in connection w[ith] electrostatic repulsion. (Goddard was to do f[ur]ther work on this as an honorary fellow at Cla[rk] University.)

(3) April 4, 1908: large mass of explosive to raise o[ne-]pound final mass to great heights; with slow p[ro]pulsion in the atmosphere. (This is, of cour[se,] germane to his work in 1919.)

(4) June 9, 1908: liquid hydrogen, oxygen, nitrog[en] tetroxide (N_2O_4), and ethane (C_2H_6).

(5) June 19, 1908: camera sent around distant plan[et] and returned to Earth.

(6) June 24, 1908: circling a planet to decrease spe[ed] before landing.

(7) October 15, 1908: steering automatically by pho[to]sensitive cells.

(8) January 24, 1909: multiple rockets. (He was to [do] further work on this while honorary fellow [at] Clark.)

(9) April 6, 1909: cooling a nozzle by liquid hydrog[en] and oxygen.

Within a year, Goddard had added such ideas as p[ro]duction of hydrogen and oxygen on the Moon, the ge[n]eral theory of the hydrogen and oxygen rocket, and means for neutralizing the effect of the decrease [of] gravity on an operator.

The six to seven years subsequent to graduation fro[m]

[1] 6 vols. 1st pub. 1854.

/PI, that is to 1915, were intense and trying for oddard. He taught and he continued graduate work. Ie obtained an A.M. from Clark University in 1910 nd his Ph.D. in physics from the same school in 1911. is doctoral dissertation was the explanation of crystal ctifiers; or, to use its formal academically heavy title: On the Conduction of Electricity at Contacts of Disnilar Solids." (It was published in the *Physical Review* June 1912.) Goddard admitted candidly that his octoral subject was chosen not because he was intersted in it. Far from it, apparently. But because his revious studies "on the conductivity of powders" at /PI would help him. He squandered nothing!

By now Goddard was combining teaching with docoral research and postdoctoral research. He was inructor of physics at WPI from 1909–1911. He then ent a year at Clark, 1911–1912, as an honorary fellow physics. From 1912–13, he was a research instructor physics in Princeton University. In 1914, he returned Clark, where he was to remain academically, as inructor and fellow in physics. In 1915 he was promoted ssistant professor and 1919 promoted professor of hysics. In eight years, not all of which he spent in resience, he had advanced from Ph.D. to full professor.

These very early years as teacher and research fellow e important to our understanding of Goddard's total areer. When at Clark in 1911–1912, he worked on varis rocketry methods, and the theory of the multipleharge rocket was roughly outlined as was the use of a ghtweight solar energy engine. (These were mentioned his notebooks.) At Princeton, he was engaged day nd night. He writes: "During the day I worked on an teresting physical problem not then explained by ectrical theory—the positive result of force on a marial di-electric carrying a displacement current." This ight be explained a little more familiarly by saying he roduced the first laboratory demonstration of meanical force from "a displacement current" in a magtic field. This is a fundamental concept in Maxwell's eory of electromagnetic waves. In the evenings, oddard worked on rocket propulsion theory. He asmed that an efficiency of 50 per cent could be secured th nitrocellulose smokeless powder and hydrogen and ygen. This theoretical work was the basis for the 1919 iblication.

Even when the schedule became too much and he was riously ill with tuberculosis in 1913–14, Goddard did t really rest and in May 1913 he wrote the material r two US Patents which he himself later said covered e essentials of rocket propulsion. They were taken out thin a week of each other in July 1914: No. 1,102,653 Rocket Apparatus (2-step Powder Rocket) and No. 103,503—Rocket Apparatus (Cartridge Reloading). It better that we allow Dr Goddard to speak of them:

> . . . They give us as nearly as possible an answer to the question as to what the "Goddard Rocket" is. . . . I prefer . . . to consider the three principles covered in the claims of these two patents, namely (1) the use of a combustion chamber and nozzle; (2) the feeding of successive portions of propellant, liquid or solid, into the combustion chamber, giving either a steady or continuous propulsive force; and (3) the use of multiple rockets, each discarded in succession as the propellant it contains is exhausted.

In the Fall of 1914 Goddard returned to Clark as a part-time teacher and fellow and immediately began further research that would, within five years, lead to his now famous 1919 Smithsonian publication. Although his entire career to 1919 was a preparation, the period 1914–1919 was particularly so. That Fall he worked out completely the theory and calculations for smokeless powder, hydrogen, and oxygen and began experiments on the efficiency of ordinary rockets. He learned that the initial mass needed to send one pound to infinity for hydrogen and oxygen, at 50 per cent efficiency, was 43.5 pounds, very close to the 45 pounds estimated on January 31, 1909.

The pace of his work at Clark was stepped up. He experimented with a vacuum tube "in which ions moved in closed paths due to a magnetic field." On May 4, 1915, he obtained a patent for "A Method of and Means for Producing Electrically Charged Particles." In September he patented an apparatus for producing gases, hydrogen and oxygen, for manufacturing liquid oxygen and hydrogen.

In 1915, Goddard began building solid propellant rockets and over the next couple of years was to spend some $800 of his own money to support these experiments. Chiefly, he was concerned with measuring the efficiency of common rockets, the efficiency of steel rockets provided with nozzles, and efficiency of steel rockets *in vacuo*. With great care, proceeding very cautiously, this work was written up. By the middle of 1916, he had reached the limit of what he could accomplish on his own resources. He began to look around for funds.

He gathered together some photographs with his notes, had them bound in a book with a green cover and a neat gold border, and called the manuscript, "A Method of Reaching Extreme Altitudes." Its aim was simple: to secure sufficient funds to permit the development and construction of a rocket with a large proportion of propellant to total weight.

It might be educational to observe Goddard's cautious exclusions and inclusions in the manuscript. For example, although measurements for projection to infinity were included, to avoid giving the impression of too much speculation, he thought it best to suggest sending a mass to the Moon rather than suggest interplanetary space navigation. He omitted the discussion of solar energy to decrease transit between planets. He omitted the discussion of the Moon as a half-way station. But he left just one tantalizing hint:

> There are, however, developments of the general method under discussion which involve a number

of important features not herein mentioned, which could lead to results of much scientific interest.

A letter apropos the manuscript was dispatched to the Aero Club of America. No reply. Other letters followed. No encouragement. Then, almost at the end of his list, Goddard sent a letter to the Smithsonian Institution. He received an encouraging reply and a request for the manuscript to be examined by a committee. It was December 1916.

In about three weeks, Goddard received a letter from Dr. Charles D. Walcott, then Secretary of the Smithsonian, commending him on the manuscript and asking him how much money was needed. With typical academic caution and wariness, Goddard debated between asking for $2500 and $10,000. He settled on $5,000. In the next letter from the Institution there was an advance, a check for $1,000, more money that he had ever seen at one time.

Then began early in 1917 the series of experiments which would, we must know, launch seriously modern rocketry and which would bring fame to Goddard. The work was carried on chiefly in the Magnetic Laboratory at Worcester Polytechnic Institute but, also, in part at Clark. Such work had no more than commenced, when World War I was declared. Goddard desired to undertake the work as a defense measure and in the winter of 1917 he obtained approval in Washington and was soon named Director of Research, U. S. Signal Corps. Until June 1918, research continued at WPI; from June until almost November it was carried on at the Mount Wilson Observatory shops in Pasadena, California.

On November 10, 1918, at the Aberdeen Proving Grounds, before representatives of the Signal Corps, U. S. Flying Corps, Army Ordnance, et al., Goddard demonstrated the results of his research. He launched an anti-tank and anti-personnel projectile rocket, about three inches in diameter, from a launching tube with two short rear legs. It was the forerunner of the bozooka of World War II. He also launched a trajectory rocket, a small multiple-charge rocket, with five charges in a combustion chamber much like a repeating rifle. It carried a simulated warhead. It travelled "straight," Goddard says, for 60 to 80 feet.

The Armistice next day put an end to Goddard's work for the Army.

Upon his return to Clark in 1919, Goddard was urged by a colleague to published his data. He turned to the Smithsonian, asked them. He correctly thought a single publication more effective than several papers—an instructive idea. The Smithsonian agreed to publication *if* cost were taken from his grant for rocket development. Goddard assented. Not quite as cautious as he was with the 1916 manuscript, he included supplementary discussions on secondary rockets, use of hydrogen and oxygen, meteoric collisions, etc.

Full bibliographic information is as follows:

Goddard, Robert H. "A Method of Reaching Extreme Altitudes." *Smithsonian Miscellaneous C[…]lections*, 71, No. 2 (May 1919), 69 pp., with […] plates. Publication Number 2540.

By now it is a collector's item. Exactly 1,750 cop[…] were published. Although dated May 26, 1919, it w[…] issued simultaneously with a news release by the Ins[…]tution on January 11, 1920. It is basically the sa[…] report as that of 1916. Goddard's experiments simp[…] corroborated his previous conclusions; only the fact[…] data based on his post-1916 tests needed revision. T[…] work had cost a total of $11,000. The paper was divid[…] into three main parts: Theory, Experiments, and Cal[…]lations Based on Theory and Experiment. There w[…] seven appendices containing supplementary discussio[…] additional notes, and 29 photographs.

Early in the paper, Goddard stresses the importa[…] of the subject and suggests a means to explore t[…] atmosphere above 20 miles. "In fact," he says, "t[…] most interesting, and in some ways the most importa[…] part of the atmosphere lies in this unexplored […]gion. . . ." Those matters which he deemed importa[…] to be investigated were density, chemical constituti[…] temperature, the aurora, and alpha, beta, and gam[…] rays. He then makes this curious statement: "[…] instruments for obtaining data at these high altitu[…] are herein discussed, but it will be at once evident th[…] their construction is a problem of small difficulty co[…]pared with the attainment of the desired altitudes."

We should accent that Goddard's prime interest w[…] this region 20 to 50 miles above Earth; we should a[…]cent that its exploration and its scientific analysis we[…] uppermost in his mind and not planetary or int[…]planetary travel.

Regarding rocket action, Goddard continues, "t[…] problem was to determine the minimum initial mass[…] an ideal rocket necessary, in order that, on continu[…] loss of mass, a final mass of one pound would rema[…] at any desired altitude." Provided that a velocity […] 7000 ft/sec could be attained and that most of t[…] rocket was propellant, to raise one final pound to […] miles required a starting weight of 3.66 pounds; […] send one pound to infinity, 602 pounds.

To hit the Moon (and with this obiter dictum Go[…]dard titillated the newspapers), a total initial mass […] 8 or 10 tons "would, without doubt, raise sufficie[…] flash powder for clear visibility." It was his plan […] send Victor flash powder to signal impact with t[…] lunar surface.

After initial experiments with black powder, […] used dense smokeless powder. Charges were fired […] steel chambers with throat diameters of about o[…] quarter and one-eighth inch and 8-degree diverge[…] nozzles with various expansion ratios. An efficiency […] over 64 per cent and an average velocity slightly und[…] 8,000 ft/sec were attained. Then to prove these v[…]locities real and knock down a prevalent unscienti[…] notion, Goddard fired the same steel chambers *in vac[…]* and observed the recoil. There was a 12 to 20 perce[…]

increase in velocity. He observed: "If . . . *successive charges* were fired in the *same chamber*, much as in a rapid fire gun, *most of the mass* of the rocket could consist of propellant. . . ." This was what Goddard termed a "step-rocket" and discussed it to some extent.

He drew an early conclusion: "It is believed that not only has a new and valuable method of reaching high altitudes been shown to be *operative in theory*, but the experiments herein described *settle all the points upon which there could be reasonable doubt.*" Then at the end of "A Method" he waxed even stronger: ". . . It remains only to perform certain necessary preliminary experiments before an apparatus can be constructed that will carry recording instruments to any desired altitude." (Italics throughout are Goddard's.)

Although one critic in the 1950's said that "A Method" "did not indicate a talent moving with ease in the field of mathematically controlled speculation," Mr. Alfred Africano (STL), who subjected the paper to a thorough analysis in 1946 and applied Goddard's calculations to the V-2, found an impressive "agreement of the actual with the calculated minimum weight ratio within 2 per cent for the first interval [5,000 ft], 1 per cent for the second [15,000 ft], and 3 per cent for the third [25,000] and fourth [45,000]."

(Intervals were Goddard's way of dividing altitude.)

Too bad, really, that Goddard was not removed to one of his intervals when the story of Publication 2540 broke on Page One of the New York *Times*, Monday, January 12, 1920.

BELIEVES ROCKET
CAN REACH MOON

This was the headline. The story related how Goddard had invented a "new type of multiple-charge, high efficiency rocket of entirely new design" to explore "the unknown regions of the upper air" and even the Moon itself. Then followed a lengthy summary of the paper in which the scientific elements were stressed, something the headline did not do. The *Times* felt that sending recording apparatus to moderate and extreme altitudes would greatly benefit meteorology and weather forecasting. Indeed, the story said boldly that a Moon shot "would be of little obvious scientific value," although of "great general interest."

The Olympian *Times* had a good thing. In "Topics of the Times," January 13, the editorial writer, under the sub-head, "A Severe Strain on Credulity," was nasty:

> . . . After the rocket quits our air and really starts on its longer journey, its flight would be neither accelerated nor maintained by the explosion of the charges it then might have left. To claim that it would be is to deny a fundamental law of dynamics, and only Dr. Einstein and his chosen dozen, so few and fit, are licensed to do that.

The tirade delivers itself of fustian:

> That Professor Goddard, with his "chair" in Clark College . . . does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react—to say that would be absurd. Of course he only seems to lack the knowledge ladled out daily in high schools.

A newspaper should know when it has a story. On January 16, the *Times* printed a letter from crusty Admiral William S. Sims, USN, who insisted that a rocket would ascend in a vacuum. He recalled a physics experiment at the Academy in 1882.

Goddard himself was aghast and wry. In the public's mind, it was a "Moon-rocket." And Goddard felt that "by trying to minimize the sensational side, I had really made more of a stir than I would if I had discussed transportation to Mars, which would probably have been considered ridiculous by the press. . . ." He took certain steps, firmly. On January 19, 1920, in the *Times*, he stated flatly, "Too much attention has been concentrated on the proposed flash powder experiment, and too little on the exploration of the atmosphere." He tried, vainly, to instill a scientific attitude in newspaper readers:

> The point is this: Whatever interesting possibilities there may be of the method that has been proposed, other than the purpose for which it was intended, no one of them could be undertaken without first exploring the atmosphere. Any rocket apparatus for great elevations must first be tested at various moderate altitudes. . . . Hence . . . an investigation of the atmosphere is the work that lies ahead.

(Need a parallel be drawn between this and the early shots at Cape Canaveral?)

After disposing, Goddard proposed raising 50 to 100 thousand dollars by popular subscription "to be used by the Smithsonian Institution in preparing for, and undertaking, a preliminary exploration of the atmosphere." Naturally, not much if anything was ever heard of the subscription again.

Although not altogether clear, it would seem that Goddard was working on a rocket that was scheduled for launch. On April 29, 1920, the *Times* carried a story that the National Geographic Society said the test was set for July, the object of which was "the study of the aurora and the short wave lengths in the spectrum of the Sun which are entirely absorbed by air." Yet the headline said:

MOON ROCKET TEST
IS SET FOR JULY

The rocket would have been propelled by explosion of successive charges. Guidance would have been obtained "through the use of photosensitive cells which, through influence of rays of light, [would] keep the rocket on its course." He was drawing on his 1908 notebook entry.

Five months later, September 20, 1920, a Page 1 box announced a test within a month. Goddard was quoted as saying that he would attempt no great altitude "for the simple reason that the funds avail-

able . . . will permit the completion of only a small model of about six pounds weight, with a capacity of about 60 charges. . . . The chief reason why the work is proceeding slowly is the lack of adequate support."

A January 28, 1921, story announced the shot for "early next Summer."

It never came off.

His complaint of inadequate support is not borne out, for he was busy all this time. In 1920 he obtained three important patents, one for "Means for Producing Electrified Jets of Gas." From 1920 to 1922, on a grant from Clark University, he performed various experiments with liquid oxygen and liquid hydrocarbons, such as gasoline, propane, and ether. These were his most important experiments—he was removing himself from the solid propellant field—and they were financed. He arrived at the conclusion that the most "practical combination" was liquid oxygen and gasoline. He literally never changed his mind. He utilized a proving stand and carried out secret experiments near Auburn, Massachusetts.

> In these experiments it was shown that a rocket chamber and nozzle . . . could use liquid oxygen together with a liquid fuel, and could exert a lifting force without danger of explosion and without damage to the chamber and nozzle. These rockets were held by springs in a testing frame, and the liquids were forced into the chamber by the pressure of a noninflammable gas.

Goddard had learned his lesson about the wayward press and withheld information for a number of years. News of "A Method" got to Germany, naturally, and in May 1922 he received a letter in quaint English from Heidelberg:

> Dear Sir
>
> Already many years I work at the problem to pass over the atmosphere of our Earth by means of a rocket. When I was now publishing the result of my examinations and calculations, I learned by the newspaper, that I am not alone in my inquiries and that you, dear Sir, have already done much important works at this sphere. In spite of my efforts, I did not succeed in getting your books about this object. Therefore I beg you, dear Sir, to let them have me. At once after coming out of my work I will be honored to send it to you, for I think that only by common work of the scholars of all nations can be solved this great problem.

A copy of "A Method" was dispatched immediately. The German mathematical student brought out at his own expense in 1923 in Münich *Die Rakete zu den Planetenäumen* (*The Rocket into Interplanetary Space*). Its author was Hermann Oberth.

The year 1923 also found Professor Goddard ready for a liquid-fuel test. On November 1, he static-fired a small rocket. It was tied down. Results were encouraging but he wasn't satisfied. "For one thing, there was the problem of getting the fuels from the tanks in the combustion chamber fast enough. He had use small pumps. . . ." So almost at once Goddard en countered a problem with liquid fuels that still is no solved altogether. He was, step by step, entering alone—the realm of systems engineering and technic direction. He had touched upon vehicle developmen and he was now into propulsion.

Yet he must have perceived success, for he went be fore the American Association for the Advancement o Science at the end of December 1923 and told th meeting at Cincinnati that he had "only one mor step to make before he could prepare a model for flight some 50 miles above Earth. (We might ask: Was tha "step" the pumps?) The rocket would carry "ver delicate apparatus" and would open and close auto matically when the rocket reached a certain altitud thereby "locking into a chamber a quantity of th air." Upon recovery of the rocket, Goddard would ex amine the specimen in his laboratory. He had som unusual ideas about the chemosphere: he was certai that frozen nitrogen would be found at 50 miles.

In addition to the air trap, Goddard's persona space physics department would measure pressur electrical effects, radiation, and temperature at variou altitudes.

The AAAS meeting was further informed that th rocket would be fired from Earth at a speed of 6 mi/se "six times faster than the speed ever attained by cannon ball." Such speed "would enable the rocket t free itself from the attraction of the Earth and keep o travelling for the desired distance." It would be pro pelled by liquid oxygen. The skin would be highl polished and marked "so that the experimente [could] keep it under observation long enough to ca culate its range and course and to estimate its landin place."

This is both amusing and admirable, in light o present-day state-of-the-art. And it is illustrative of strange quirk: a desire for publicity, after all. It is ver doubtful he seriously believed in such a speed.

On New Year's Day, 1924, having returned to Wor cester, Goddard delivered himself of a familiar plaint "Work on the high altitude rocket must be supporte and models supplied for actual trial flight during th coming year, if America is to continue her lead in thi branch of scientific research."

Perhaps either to encourage such support or t solicit it, Goddard allowed himself to be the subject o a Sunday feature in the New York *Times*, May 25 1924. Under the large headline, "HOPES TO REACH MOON WITH A GIANT ROCKET," was a lon article about him and his rocket. However, now tha he knew the methods of the press, he was very cautiou to discuss little but his "giant" experimental rocket 60 inches long and 6 inches in circumference.

An implied comparison with M. Jules Verne failed t elicit any "wild blue yonder" comments. Instead, God

dard discussed propulsion: "successive explosions" vs "regular propulsion, supplied by gases released in a steady flow." He favored the latter. His scientific manner is well put:

> I am working with well-known gases, but have found new ways of applying their use. The experimental rocket also will have considerable mechanism. I need not say that it has required years to study the many details involved. Almost every step has been an experiment within itself. But I have proved . . . that the rocket will rise practically any desired distance. This being true, there is nothing to prevent the construction of a greater rocket which would go up to the Moon.

This set off the reporter and he again switched to Verne and his cannon. He tried to inveigle Goddard into discussing a "space car," but his effort is wasted.

Nevertheless, realization that publicity was not without its value must have led Goddard to allow President W. W. Atwood, Clark University, to announce at commencement exercises in June "the solution of the problem of an explosive for the Goddard rocket. A liquid will be used to give propelling force without overheating the projectile." Atwood added that a model of the rocket was complete and would be sent up later that Summer. It was never fired.

During these early years of work on liquid-fueled rockets, the Smithsonian Institution, under Dr. Charles G. Abbott, gave Goddard grants; as a matter of fact, from 1922 to 1930. The Carnegie Institution, as well, under the direction of John C. Merriam, was interested in the work and arranged for a substantial financial contribution to Clark. The school, also, added its own aid.

Firmly battened by such a combination, Professor Goddard was able to prepare for the signal moment of his career, March 19, 1926.

There was another moment, a very unscientific moment, we must not ignore—his marriage on June 21, 1924, to Esther Christine Kisk of Worcester. A well educated woman she was, having been graduated from Johns Hopkins. She met Goddard at Clark and became his secretary. After marriage, she was present at almost every rocket launch and she was the "official" photographer.

Late in 1925, the New York *Times* was tracking Goddard again. They ran a story about the fated young Austrian pilot-astronomer-rocketeer, Max Valier, and his plans to reach the Moon by rocket travel. The story appeared in the Sunday Magazine Section, November 22 and was 90 per cent concerned with Valier; however, at the very end Goddard was interviewed. Instead of praise or enthusiasm, he showed stiffness and talked very coldly about experiments. He probably had on his mind the worries with his liquid-fuel static tests. He was ready to try again after almost two year's preparation. He felt he had overcome the problem of the pumps. In December he tested his second rocket. The fuels were forced "into the chamber by the pressure of . . . nitrogen." It worked well.

There remained but to fly it.

March 16, 1926, was cold. Goddard had no classes that day. There were two inches of snow on the ground and the ground was hard and the grass stubbly. Goddard, his wife, and his assistant, Henry Sachs, loaded the equipment on a trailer and then climbed in a roadster with the isinglass curtains up. They drove quietly and quickly to Aunt Effie Ward's farm near Auburn, Massachusetts. She didn't mind—anything that Robert did was all right with her.

Goddard and Sachs set up the launching stand and attached the rocket. Both men posed by their creation, bundled against the cold. Goddard wore a cap, overcoat, gloves, and buckled overshoes. Sachs simulated igniting the rocket. He wore ear-muffs. Esther snapped the pictures. The land was bleak and the snow covered the base of the launching stand.

The rocket on the stand was queer looking. The fuel tanks were "slender tubes, placed one behind the other." The combustion chamber, about the size and shape of a 4th of July rocket, and exhaust nozzle were well ahead—the opposite of today's design.—and supported on "spidery arms which also carried the fuel lines." Overall length was perhaps 10 feet, about half of which was the rocket proper. "Pressure to force the fuels into the combustion chamber was furnished by an outside pressure tank and, after launching, by an alcohol heater carried on the rocket."

A mistaken idea that this was improvised has managed to grow. Actually the design was carefully planned, and Goddard thought it advantageous in keeping the flame away from the tanks and in producing stabilization. For the latter it was of no value. "This is evident," Goddard later commented, "from the fact that the direction of the propelling force lay along the axis of the rocket, and not in the direction in which it was intended the rocket should travel, the condition therefore being the same as that in which the chamber is at the rear of the rocket."

Henry Sachs had his blow torch going. It was attached to the end of a long pole. He stood off to one side and touched the flame to the igniter at the very tip of the rocket motor. Esther turned the crank of her camera to record the action. Flame shot from the nozzle. The rocket rose 41 feet, turned over, travelled 184 feet, reached a speed of 60 miles per hour, and stayed in the air a total of $2\frac{1}{2}$ seconds.

In the cabbage patch where it had smashed, the three of them picked up every smidgen. They had just conducted, successfully and secretly, the first liquid-fuel rocket flight in the world. But they were Yankees, too, and they saved the pieces.

Not one whisper reached a newspaper.

Apparently in the next two years, Goddard fired off a number of similar rockets; however, the information is sketchy and the designs and results are unknown.

He was quite busy, at times, with newspaper and periodical publicity during this time. On July 4, 1927, the New York *Times* published a story about an "ocean rocket" of his that would carry passengers across the Atlantic at "terrific speeds," much faster than Byrd or Lindbergh. Goddard played down its immediacy and said that there were many details to be worked out. (As a matter of fact, it was not until 1932 before his final thoughts on the idea were made known.) Another *Times* story must have given him some small satisfaction, as a scientist. A Sunday magazine story carried the huge headline: "A ROCKET AUTO OPENS VISTAS OF STAR VOYAGES." More important, especially to Goddard, was the subhead, which was central to his ideas and preachments: "But First Rocket Planes Must Test Air Many Miles Up and Leap the Atlantic in an Hour and a Half—A Principle of Vast Possibilities." Goddard was cited in the lengthy piece, along with Valier, Opel, Oberth, and Esnault-Pelterie.

In the Summer of 1928, *Scientific American* called on his good services to comment on an article that posited a flight from Earth to Mars, propelled by smokeless powder, in 7 months. He did not really commend the piece, but by an odd circumlocution came up with this statement:

> According to recent press notices, interplanetary transportation must remain impossible until atomic energy can be obtained and controlled. This attitude is . . . much like that of the scientists of thirty years ago, who declared that an airplane could not operate unless the force of gravity could be neutralized.
>
> If atomic energy were available, it would be a very convenient means of propelling an interplanetary rocket. Atomic energy is not, however, necessary, as an interplanetary flight is possible with means even now at our disposal. This is set forth in my article in . . . 1919. . . . I can say that, although [the] article . . . may read like romance, it is nevertheless thoroughly scientific, and, while not telling the whole story, it gives a good picture of what an interplanetary rocket must be like.

We come, then, to July 17, 1929.

The *Times* felt it good enough news next day to run a story on Page 2: "METEOR-LIKE ROCKET STARTLES WORCESTER." Goddard called the shot "Test of Medium-Sized Rocket Having High Center of Gravity and Low Center of Air Resistance"—a singularly unimpressive title. The scene was again Effie Ward's farm. But the equipment was far more elaborate. A 40-foot tower (also reported as 60-foot) stood in a shallow gulley. An $11\frac{1}{2}$-foot rocket, fitted on two $\frac{3}{8}$-inch rails, rested vertically within the frame of the tower. The combustion chamber was located at the rear of the rocket. It was the best location, Goddard was now convinced, because "no part of the rocket" was in the "high velocity stream of ejected gases, and none of the gases" was directed at an angle with the rocket axis. Up a slight rise, perhaps 60 to 75 feet away, stood a wooden shack. Inside a small group of experimenters were arranged around an electric detonator. The rocket, which weighed 32 pounds empty, held 14 pounds of gasoline and 11 pounds of liquid oxygen. It also carried an aneroid barometer, a thermometer, and a camera which was focused on both instruments. It was to be operated by a trip lever at the ejection of a parachute. Everything was ready.

Someone in the wooden shack pressed an electric switch.

What happened in the next few seconds is not entirely clear.

The rocket shot upward with a tremendous roar; just how high is disputed. Anywhere from 90 to 300 feet. Near apogee it exploded violently. The noise was heard for two miles around. The townspeople of Worcester were frightened beyond reason. Even as a parachute carried the empty shell of the rocket gently to Earth, the people went into action. At first, they thought a huge meteor had exploded. Other witnesses insisted that an airplane in flames had flashed across the horizon and blown up. Two police ambulances raced to the scene, "looking for the supposed victims." An airplane took off from the local field and submitted the environs to reconnaissance. Newspaper men appeared.

Goddard pleaded that they not publicize the event. But freedom of the press will out when to its advantage, and of course his pleas were ignored. It was too good a story. "Moony" Goddard was at it again. Even before they arrived home, Mrs. Goddard recalls, extras were coming out. The professor was forced to issue a statement.

> The test this afternoon was one of a long series of experiments with rockets using entirely new propellants. There was no attempt to reach the Moon or anything of such a spectacular nature. The rocket is normally noisy, possibly enough to attract considerable attention. The test was thoroughly satisfactory; nothing exploded in the air, and there was no damage except incident to landing.

He did not really explain the noise—something which everyone seems to have been agreed upon. Regardless, America's second liquid-fueled rocket had been fired publicly and the newspapers were pleased to call it a "Moon" rocket.

And governmental officials were pleased to ask Goddard to explain. He called on the Massachusetts state fire marshal. Dissatisfied, the marshal withheld permission to conduct further tests. Goddard then called on the military authorities at Camp Devens, 2[illegible] miles north of Worcester, and asked them for permission to carry on his field firings. (He had used the camp once before.) The military were afraid of fires and demurred. Determined, the professor appealed to the War Department and was supported by his faithful Dr. C. G. Abbott of the Smithsonian. On October 2[illegible] 1929, the War Department granted permission to use an abandoned farm near the artillery range of Camp Devens, said permission subject to the understanding

that Goddard would take precautions against fires. Flights were permitted only after a rain or when snow was on the ground.

The tower at Auburn was broken down and transported to the Camp. Another precaution was a sheet-iron windshield erected on three sides of the tower. A shed 50 feet in front of the tower protected the personnel. The first static test took place on December 3.

Meanwhile, a series of newspaper and magazine articles on Goddard more or less culminated in an article in the Sunday New York *Times*, October 13, 1929, in which the professor was revealed to be working on a solar or Sun motor. However, the article went on to say, "there is still no method of turning captured solar energy efficiently into propulsive energy, for rocket propulsion requires that some of the mass of the flying craft be left behind in space as well as some of the energy, this discarded mass providing the 'action' to which the 'reaction' of the rocket is the twin."

The conclusion was familiar. "Passenger rockets are little likely, however, to be the next use of rocket craft. Professor Goddard disclaims immediate interest in this passenger problem. What he wants is a rocket able to pierce most of the Earth's blanket of air." *Popular Science Monthly*, in an editorial in October, had lauded Goddard for such an interest: "While others have talked of shooting rockets to the Moon, Professor Goddard's success thus far proves him the most practical of the lot."

The *Times* story was read by Colonel Charles Lindbergh, then riding the top of a fantastic wave of hero worship. He was at the time "particularly interested in any research that would provide accurate information about the upper layers of the atmosphere." On a Sunday afternoon in November, the 24th, the Colonel slipped into Worcester and visited Goddard's laboratory for two hours. The professor outlined his work in progress to the young flier and ran motion pictures of his rocket flights. Lindbergh deliberately avoided reporters on his way out of the city, and Goddard refused to comment on the visit.

But Lindbergh was probably the most important guest of Goddard's life, for out of the visit to the laboratory were to come grants and interest from the Guggenheim family during the 1930's. However, that is another story for another article.

Shortly after the static tests got underway at Camp Devens, the New York *Times* (indefatigable, when seeking after Goddard) ran yet another long story about him and his rockets on Sunday, December 15, 1929. Written by R. L. Duffus, it was a full page long and was the most thorough and accurate account yet printed. Both the headline and the story emphasized that "this Winter" a "great rocket" would be fired to explore outer space. The tone was sober; the information was wide. About the only drawback are three incredibly poor illustrations—and the fact that the rocket was not fired.

Before we leave Goddard and his achievements up to 1930, we must discuss another aspect of the man which we haven't had time to consider—his inventions. Goddard was an inventor and an active one. In *Popular Science Monthly*, October 1929, he wrote an article describing and analyzing a rather unusual device: a solar motor. Although dated now and even antique, it is of some interest. Goddard claimed an efficiency for it of at least 50 per cent, "more than twice that of the finest steam turbine electric generating plant in operation today."

In explaining its operation, Goddard pointed to the fact that "instead of the usual kind of boiler, a hemispherical end piece, made of clear fused quartz, is bolted to a hollow body. Water is pumped in and mercury under high pressure is sprayed into the water. . . ." Quartz would not crack even when subjected to enormous temperatures. This hemispherical end piece was suspended from the apex of three Duralumin struts attached to a large (about 20 feet) parabolic mirror which focused Sun rays on the quartz "so that the point of greatest heat was where the spray of mercury formed a screen in the water."

Let Goddard continue in his own way:

> The water itself, being transparent, will not absorb the heat, but the mixture, being opaque to light, will absorb all the heat instantly. The resulting high temperature will convert the metallic mercury into mercury vapor at a correspondingly high pressure, and the water will be converted into steam at high pressure.

There were at least two advantages to this design: (1) the quartz could be shaped so that every ray of light would strike it at right angles and (2) the hottest point was inside the boiler instead of outside.

Goddard believed that the commercial possibilities were enormous, especially on farms. (This was, recall, before the New Deal and Rural Electrification.) A 100-foot mirror would produce at least 650 horsepower. He also considered its use in aerial navigation as well space travel. He wrote:

> Recently another possible use . . . has been mentioned in connection with my experiments in . . . interplanetary navigation which the French have named "astronautics."
>
> In 1919 I suggested the possibility of producing a rocket so powerful that it would leave the Earth's surface never to return. Since then there has been considerable speculation regarding the use of such a rocket in . . . interplanetary space. It is in this connection that I have been interested in developing a very light and efficient solar engine such as now has been designed.

He envisioned a new propulsion. He was entering upon an exciting period. The static tests at Devens had commenced. The entire Roswell, New Mexico, operation lay ahead. He had 15 years of life left him and much to do.

Alignment Chart for Satellite Orbit Calculations

Leith Holloway*

In order to track artificial earth satellites during the first few days after launch time when accurate machine-derived ephemerides are not yet available, it is helpful to be able to compute the theoretical values for the rates of motion of the nodes and perigee of the orbit and to determine the satellite's mean height from its period or vice versa. First approximation formulas for computing these quantities are given in Reference (1), and convenient transformations of them are given below. However, these formulas are awkward to use because of non-integral powers of quantities appearing in them. Therefore an alignment chart such as the one published here is a very convenient means for evaluating these formulas. The results obtained from the chart are probably as accurate as the necessary approximations justify. See Figure 1.

Because of the nonspherical shape of the equipotential surfaces of the earth's gravitational field resulting from the earth's oblateness, the nodes and perigee of a satellite's orbit are continuously changing their positions. These motions are akin to the precession of a gyroscope.

With inclination angles less than 90° the nodes of a satellite orbit advance westward with respect to the stars, or in the direction of decreasing right ascension. With inclinations greater than 90° (retrograde orbits) the nodal motion is eastward. The perigee of an orbit having an inclination angle of less than 63.4° moves in the direction of the satellite's motion, whereas above this critical angle the direction of the motion is the reverse. The alignment chart estimates these motions by solving the following equations:

$$P = 84.49\ a^{1.5} R_E^{-1.5}$$

$$\dot{\Omega} = -3.1208 \times 10^5\ (\cos i) P^{-7/3}$$

$$\dot{\omega} = 3.1208 \times 10^5\ (2 - 2.5 \sin^2 i) P^{-7/3},$$

where P is the anomalistic period of the satellite in minutes; R_E is the equatorial radius of the earth; a, the semimajor axis of the satellite's elliptical orbit; i, the inclination angle of the orbit; and $\dot{\Omega}$ and $\dot{\omega}$ are the rates of motion of the nodes and perigee respectively in degrees per day where positive results indicate motion in the direction of the earth's rotation for the nodes and in the direction of satellite motion for the perigee. The mean height is equal to a minus R_E.

In the derivation of the constants in these formulas, R_E was taken to be 3963.3 statute miles and J, the coefficient in the earth's gravitational potential function, was assumed to be exactly 0.0016238, a value recently determined in Reference (2). The magnitude of J is related to the degree of flattening of the earth at the poles.

Although the above formulas for $\dot{\Omega}$ and $\dot{\omega}$ are valid only for circular orbits, the effect of ellipticity is small with low eccentricities, amounting to only two per cent for an eccentricity of one-tenth. The theoretical value of the correction factor is the square of the reciprocal of $(1 - e^2)$ where e is the eccentricity.

* U.S. Weather Bureau, Washington D.C.

An example will illustrate the use of the alignment cha Assume that a satellite has a period of 100 minutes, an i clination of 55°, and an eccentricity of 0.12. The left sca indicates that this period corresponds to a mean height 473 miles. Lay a straight edge from this point on the left sca through the fifty-five-degree mark on the left side of t center scale and extending to the long right scale. Read t first estimate of the nodal motion on this right scale; this 3.86 degrees per day or 19.4 minutes of local time earlier ea day as read from the left side of this same scale. This tir scale is merely the negative of the angular motion of the nod converted to minutes plus 3.94, the daily motion of the s in minutes. The small correction for the eccentricity of 0.12 determined from the short scale on the far right of the cha The scalar distance given by this short scale corresponds the magnitude of the correction to the nodal motion given the long right scale. This distance should be laid off on t long right scale upward from the first estimate to the co rected value of 3.97 degrees per day. The same procedure used to obtain the perigee motion except that the fifty-fiv degree mark on the right side of the center scale is employ instead. This gives values of 2.17 and 2.23 degrees per d before and after correction respectively.

It is impractical to extend the scale for the inclinati angle used in computing the perigee motion above 60° becau of the reversal in direction of this motion at an inclination 63.4°. Rates of motion of the perigee of high-inclination orb must be computed by hand using the formula. In the spec case of a polar orbit ($i = 90°$) the magnitude of the perig motion is one fourth of what its value would be for an equ torial satellite with the same mean height and orbit ecce tricity.

There is an obvious connection between the time scale the chart and the *prime sweep interval* which is the time r quired for the orbit to make one apparent revolution arou the earth. Since the orbit is generally precessing towards t west against the rotation of the earth, the prime sweep interv is most often less than one day. In fact, it is usually express as one day minus a certain number of minutes. One day min the value obtained from the alignment chart's time scale approximately equal to the prime sweep interval. Howev the number of minutes to be subtracted will always be t large by about one per cent at 15 minutes and by about t per cent at 30 minutes. The reason for this discrepancy that the chart computes the daily rate of the orbit's local tir change, whereas the subtrahend of the prime sweep interv expresses this change in units of local time change per peri equal to the prime sweep interval itself.

Clearly the procedure of specifying the position of the or in respect to a coordinate system rotating with the mean s has obvious advantages. Knowing the local times under t orbit at a given latitude greatly simplifies the problem of fo casting the time when the orbit will enter or leave the earth shadow at this particular latitude, and it also facilitates p dicting when the orbit will be favorably located with respe to the twilight zones. This is the justification for including t time scale on the chart opposite the angular velocity scale nodal motion. Incidentally, this time scale, of course, has meaning with reference to the perigee motion and so shou be disregarded in this case.

The author is indebted to all of the authors of Referen (2) below for their kind assistance in the preparation of t chart and to Mr. Earl Rayfield for assisting with the drafti

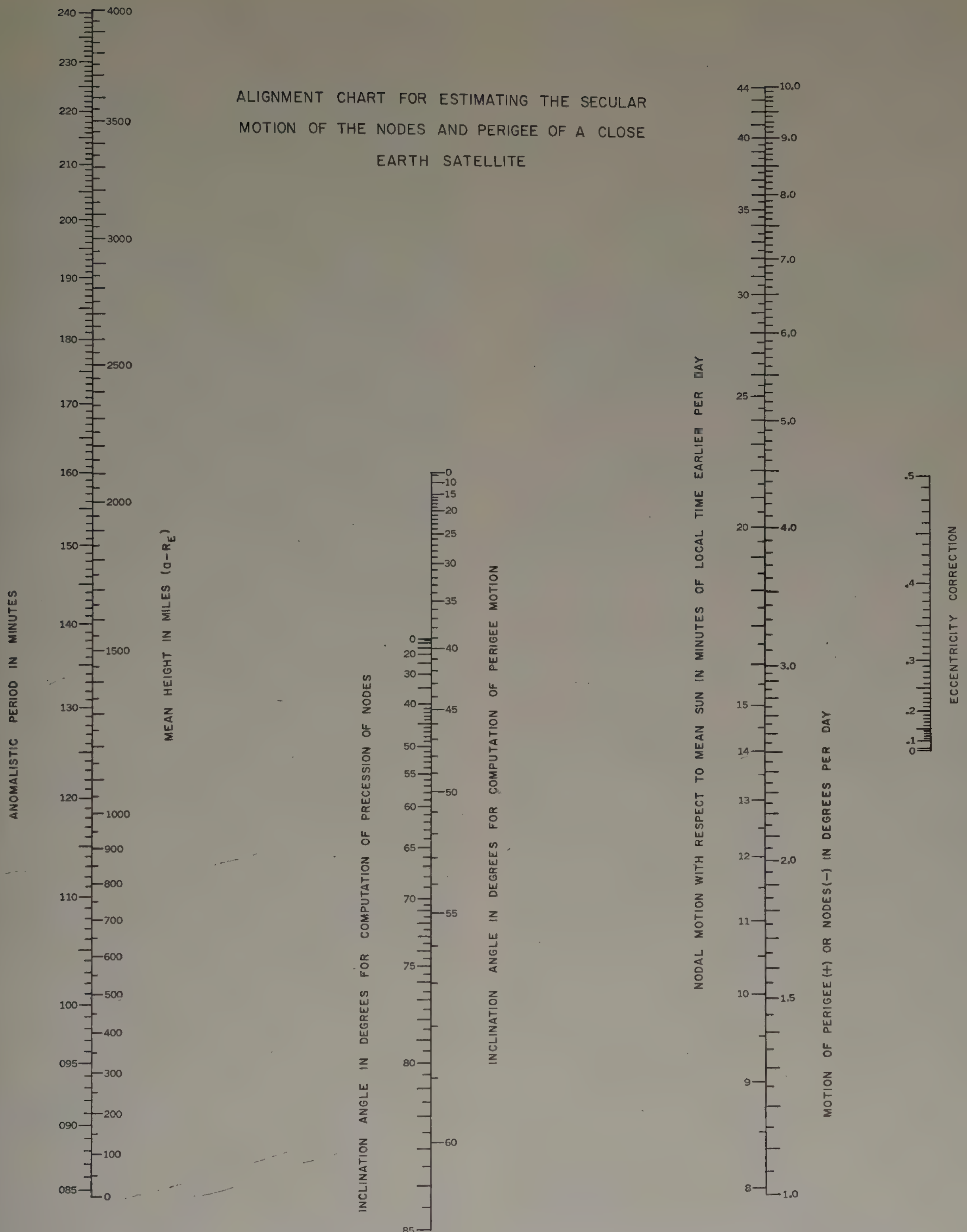

FIGURE 1

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Radiation Shelters For Lunar Exploration

Donald H. Robey

Abstract

The hazards to human life from solar flare and cosmic ray particles have been briefly considered. It is concluded that human explorers should prepare emergency flare shelters with thicknesses of the order of 50 to 100 or more centimeters of Moon material. This is assuming a specific gravity of roughly two for the surface material. Reasons are also presented for suggesting that the dark side of the Moon may not be entirely safe from flare particle radiation.

Introduction

Although the actual, manned lunar station is several years away, plans are being made for this gigantic undertaking at the present time. It is therefore not too early to speculate on the nature of shelters required for station personnel. The period of time that they remain at the base depends upon the radiation environment, the shielding, and the allowable radiation dose.

There are two well known active sources of radiation: (1) the galaxy and (2) the Sun. The Earth may even be a minor source of particle radiation. For example, secondary neutrons from cosmic primaries may leave the atmosphere in all directions.

The galaxy is believed to be the source of cosmic radiation. About 80 per cent of this radiation is believed to consist of protons, 19 per cent of alpha particles (helium nuclei) and the remaining 1 per cent of heavier nuclei such as carbon, nitrogen, oxygen and iron. Electrons are also believed to be present.

Since cosmic rays are charged, they cannot pass undisturbed through the Earth's magnetic field unless they are traveling parallel to the flux lines. The north and south geomagnetic poles have 90° dips at 76° N, 102° W and 68° S, 145° E, respectively. Particles incident vertically near these poles are not deflected significantly. Fortunately, the atmosphere acts as a radiation shield. In order for a proton to reach sea level, it must have a kinetic energy of about 2 Bev in the polar regions. However, over the geomagnetic equator, because of the Earth's magnetic field, the minimum required kinetic energy is about 15 Bev. Thus, the equatorial magnetic field is equivalent to about 6 atmospheres. Protons in the 60 Bev range, however, can penetrate the atmosphere, to sea level, at all angles of incidence.

On the surfaces of the Earth, the atmosphere provides us with about 1023 g/cm² of shielding against incident particle radiation. If the bulk density of the Moon's surface is taken to be about 2 g/cm², then roughly 511 cm, or more (since the range of protons in maria, or rock, for example will be greater than in air) of moon material will be needed in place of the Earth's atmosphere. The ratio of ranges for proton penetration in almost all of the elements (excluding hydrogen) is less than two, so that a more conservative estimate might require up to 1023 cm ($\approx$ 34 ft) of moon for the depth of a lunar surface shield.

The additional protection offered by the Earth's magnetic field except for over the polar regions has not been included. This could imply an additional 30 or 40 meters of moon.

The cosmic ray background is believed to be fairly steady except for the Forbush intensity decreases which are associated with transient activity on the Sun, i.e., solar flares. Since a decrease in cosmic ray background is nothing to be alarmed at, it is probably feasible to estimate the maximum radiation dose from this source in advance.

A fairly serious hazard from cosmic rays is the possible injurious effects on the brain and eyes, especially from heavy primaries. According to Davis (1960), some radiobiologists believe that single, heavy cosmic particle could incapacitate a man if it struck in a critical brain area. He estimates that the particles might damage a region of tissue with a diameter equal to about 20 blood corpuscles. If this turns out to be the case it is a serious blow to the manned exploration of the Moon's surface, and to space flight in general. It is probably too early to properly evaluate this potential hazard at the present time. Also the brain can repair itself or use other channels for many kinds of injuries.

If we take an optimistic attitude and assume that a man could be exposed to this radiation for several days at a time and have a good chance of not suffering serious damage, he still must cope with solar flare particles. It is known from satellite and balloon studies that at least some of the largest class 3+ solar flares eject beams of protons, helium nuclei, and electrons, in quantities that are lethal to unshielded man. The intensities may be as high as 10^3 or 10^4 times the normal cosmic ray background for periods of several hours. The energies are ordinarily below the cosmic ray range, i.e., below a few hundred Mev, but the flux densities can be orders of magnitude greater.

However, on eight occasions during the last two decades the Sun has erupted and thrown out a burst of high-energy particles that might properly be called "solar-cosmic rays." The largest of these occurred on February 23, 1956 and accelerated an estimated 10^{33} particles from thermal energies to energies of over 2 Bev (Parker 1957).

Since the time of this last great flare, approximately 25 additional events of a similar, but less intense, nature have occurred. The particles are believed to consist largely of protons. Furthermore, since the energies and flux densities of these particles can be dangerously high for many hours following flares, it will be necessary to introduce emergency shields from Moon explorers.

When on a field trip several hours distance from a base, it might be necessary to dig deep holes in the Moon for flare shelters. Boards could be carried along to support vehicles and other available equipment over openings to shelters.

It was estimated (Robey, 1960) that the large flare of May 10, 1959 would have delivered a radiation dose of about 8000 rem (based on a relative biological effectiveness of two for protons below 40 Mev) to an unshielded man in space at the Earth's distances. The dose to a man on the Moon's surface is about half of this value, if the majority of the particles are omnidirectional.

Actually there is no evidence to suggest that flare particles travel in straight paths to the Earth. In the first place, they are charged and may be forced to depart from the Sun in corkscrew orbits depending on the shape of the solar magnetic field. In addition to the possible solar dipole field there are reasons to believe (Alfven, 1954) that magnetically polarized plasmas are ejected from the Sun. Events of this kind tend to create magnetic flux tubes which curve out away from the Sun, because of its rotation, and clear the way for subsequent charged particles. Alfven (1954) postulated that fluxes as high as 10^{-4} gauss may exist in these regions at the Earth's distance. This was subsequently verified by measurements made by Pioneer V. Charged particles which follow the flux lines in these tubes might tend to be confined, which could explain why there does not seem to be a gradual buildup of the particles arriving at the Earth. That is, there seems to be a sharp leading edge.

On the other hand the decay of the particle influx is rela

ively slow, sometimes lasting for several days. The lifetime of the flare, as indicated by the intense emission of hydrogen alpha light, ordinarily does not exceed a few hours. If this also corresponds to the duration of the period of intense particle ejection, then the particles must be stored somewhere. Winckler (1959) states that a storage in the vicinity of the Sun fits the experimental facts very well, that is, the flare particles observed terrestrially, decay slowly, and stream in over the poles with high intensities both prior to and following the associated Forbush decrease in cosmic ray intensity. The sudden shock-like arrival of particles might be explained by the tendency to shove and carry along quantities of ionized interplanetary ions. This is analogous to the action of the solar wind on ionized nitrogen in comet tails. The tails are accelerated and blown out from the Sun with forces greatly exceeding the radiation pressure of sunlight.

Meyer, Parker and Simpson (1956) argue in favor a field-free ($\leqq 10^{-6}$ gauss) heliocentric cavity extending out to about 1.4 astronomical units (AU's). The lack of significant magnetic field in this region can be used to explain the sharp onset of cosmic noise absorption over the poles (attributable to the arrival of protons, which increases the ionization in the D-layer and, consequently, attenuates radio noise from space). There is evidence to suggest that the flux is omnidirectional. Both this circumstance as well as the observed slow decays in intensity can be explained by a scattering and diffusing property of the boundary of the heliocentric cavity. The boundary, which is actually apt to be a spheroidal shell, is probably several AU's thick. That is, it is probably in a turbulent or disordered state in a shell of thickness comparable to the orbital distances of Mars and Jupiter. The Sun is believed to create this cavity by the continuous action of the solar wind (Parker, 1958). Also the occasional ejections of magnetically polarized plasmas are probably able to modulate the magnetic properties in the boundary layer (between Mars and Jupiter) and thereby bring on Forbush decreases.

The implication of a heliocentric cavity is that the flare particles can scatter back and forth in the solar system and produce an omnidirectional flux. Consequently, a colony on the dark side of the Moon would not be safe from this radiation.

If the particles leave the Sun in a horn shaped beam, as suggested in Fig. 1, and are scattered at the interstellar magnetic field boundary, it is reasonable to suppose that there will be a tendency for them to remain trapped within this disturbed region created by the original beam. This prevents them from scattering immediately all over the inner solar system. Particles with higher energies are expected to diffuse out of the beam at faster rates.

If the particles were not restricted by some mechanism such as this, flares on the back side of the Sun would produce effects measurable at the Earth, e.g., cosmic noise absorption over the magnetic poles. However, the flux rate of growth would be gradual and probably less intense by an order of magnitude, rather than a sharp spike which builds up in minutes. We would expect the number of gradual buildups to be comparable to the number of sharp increases. However, the gradual buildups apparently are not observed except for rare cases, where large flares occur within two or three days after the region passes over the west limb.

Summary

Since the Moon lacks a natural radiation shield such as an atmosphere, or a significant magnetic field, it will be necessary for human explorers to have radiation shelters.

It is suggested that the living quarters and working areas

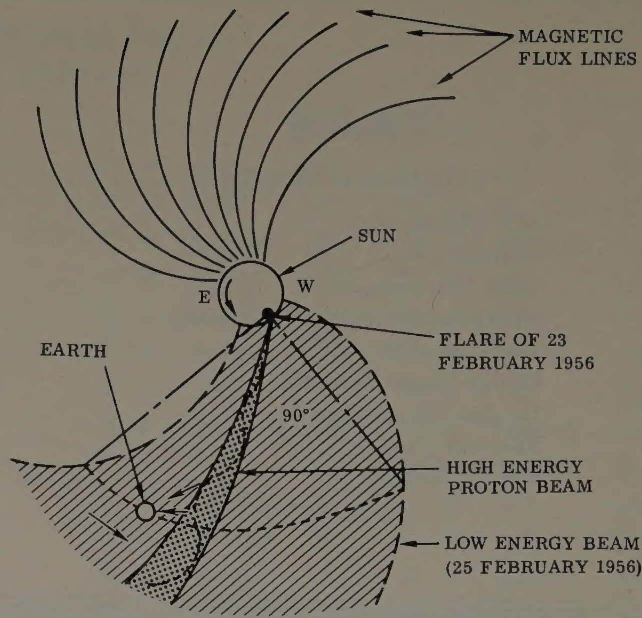

Fig. 1 showing the tendency for solar plasmas to follow magnetic flux lines. The high energy proton beam from the flare of 2/23/1956 may have missed the Earth. The first protons to arrive at the Earth apparently leaked out of the beam as indicated by arrows.

be located beneath the surface, or inside a mountain, so that at least 1023 gm/cm² of material is between the personnel and the incoming radiation. This roughly duplicates the shielding from the Earth's atmosphere but neglects the additional protection from the Earth's magnetic field. However, there is only about a 10 per cent difference in cosmic ray background at sea level in going from the geomagnetic equator to a geomagnetic latitude of about 40°. The changes are small thereafter.

When making manned explorations on the surface it is advisable to have emergency shelters available because of the hazard from solar flares. Directed, rocket-borne explosives, e.g., shaped charges, might be used to create hole shaped shelters. About 50 or 100 cm of Moon material would probably shield most flares. The personnel might have to remain in the shelter for several hours. The rarer, cosmic ray-type flares, may require two or three meters of Moon.

Reasons are given for expecting an approximately isotropic distribution of the particle influx on the Moon. It is concluded that explorers on the dark side will therefore also be in danger from solar flare radiation.

A very small, but extensive, lunar magnetic field might offer some protection.

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Α	α	alpha	(a)	Ν	ν	nu	(n)
Β	β	beta	(b)	Ξ	ξ	xi	(x)
Γ	γ	gamma	(g)	Ο	o	omicron	(ŏ)
Δ	δ	delta	(d)	Π	π	pi	(p)
Ε	ϵ	epsilon	(ĕ)	Ρ	ρ	rho	(r)
Ζ	ζ	zeta	(z)	Σ	σ ς	sigma	(s)
Η	η	eta	(ē)	Τ	τ	tau	(t)
Θ	θ	theta	(th)	Υ	υ	upsilon	(u)
Ι	ι	iota	(i)	Φ	$\phi\varphi$	phi	(ph)
Κ	κ	kappa	(k)	Χ	χ	chi	(ch)
Λ	λ	lambda	(l)	Ψ	ψ	psi	(ps)
Μ	μ	mu	(m)	Ω	ω	omega	(ō)

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a = semi-major axis
e = eccentricity
☊ = longitude of the ascending node
i = inclination to plane of the ecliptic
ω = longitude of perihelion measured from the node
T = time of perihelion passage

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$$\frac{\cos(\pi x/2b)}{\cos(\pi\alpha/2b)}$$

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$(a + bx) \cos t$ is preferable to $\cos t\,(a + bx)$

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$$\{[a + (b + cx)^n] \cos ky\}^2$$

is required rather than $((a + (b + cx)^n) \cos ky)^2$.

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